Module 2: Portfolio Theory

(BUSFIN 4221 - Investments)

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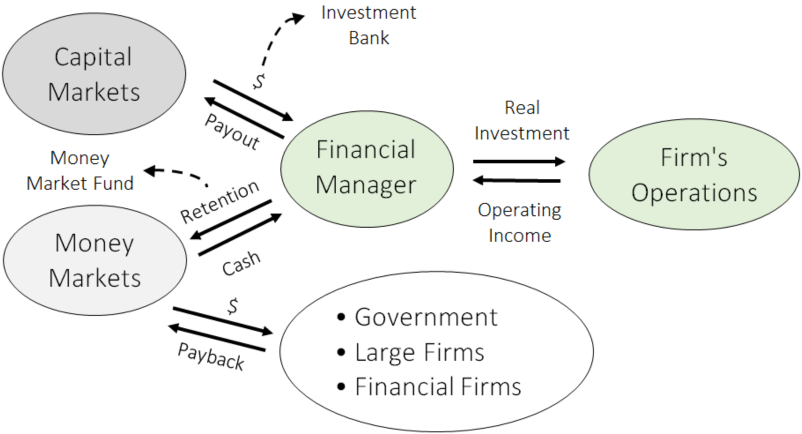
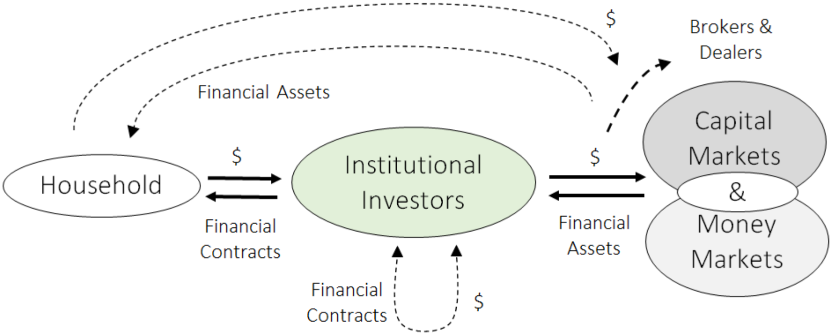
Module 1 - The Demand for Capital



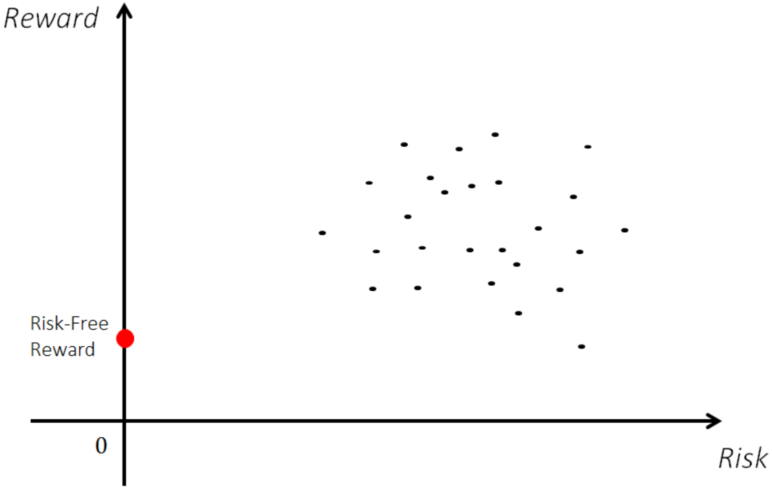
Module 1 - The Supply of Capital



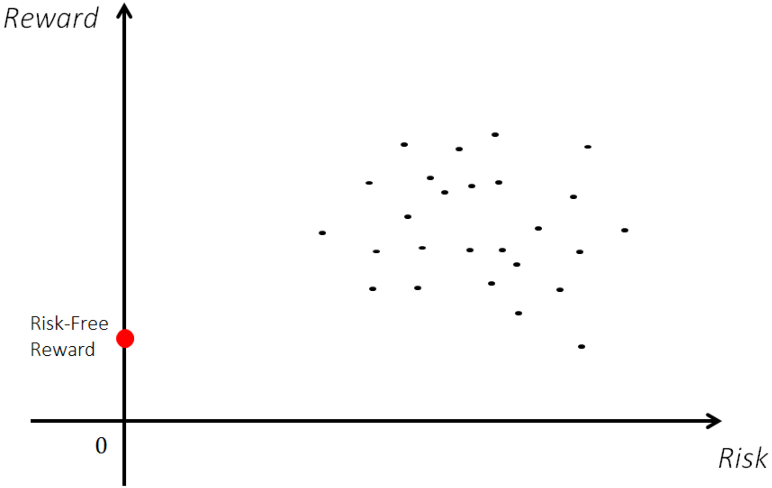
Module 1 - Investment Principle

$$PV_t = \sum_{h=1}^{\infty} \frac{\mathbb{E}_t\left[CF_{t+h}\right]}{\left(1 + dr_{t,h}\right)^h}$$

This Module: Creating a Portfolio



This Section: Defining Risk and Reward



Measuring Performance: Returns

$$r_t = \frac{(P_t + CF_t) - P_{t-1}}{P_{t-1}}$$

$$= \underbrace{\frac{(P_t - P_{t-1})}{P_{t-1}}}_{Capital\ Gain} + \underbrace{\frac{CF_t}{P_{t-1}}}_{Yied}$$

- Arithmetic average return (or simple "average return"):

$$\bar{r} = \frac{r_1 + r_2 + ... + r_T}{T}$$

- Geometric average return:

$$\bar{r}_G = \{ (1 + r_1) \times (1 + r_2) \times ... \times (1 + r_T) \}^{1/T}$$

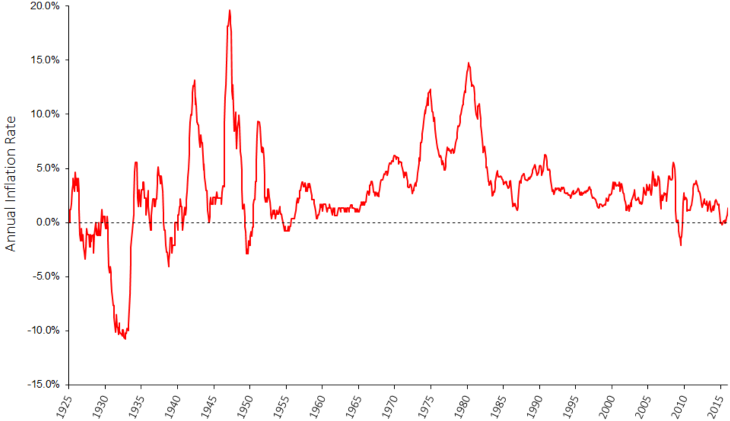
Measuring Performance: Annual Returns

- A return of 1% in the previous month is not comparable with a return of 12% in the previous year. We need to fix the time period of different returns to make them comparable
- The effective annual rate, ear_t, does that for you:

$$1 + ear_t = (1 + r_t)^n$$

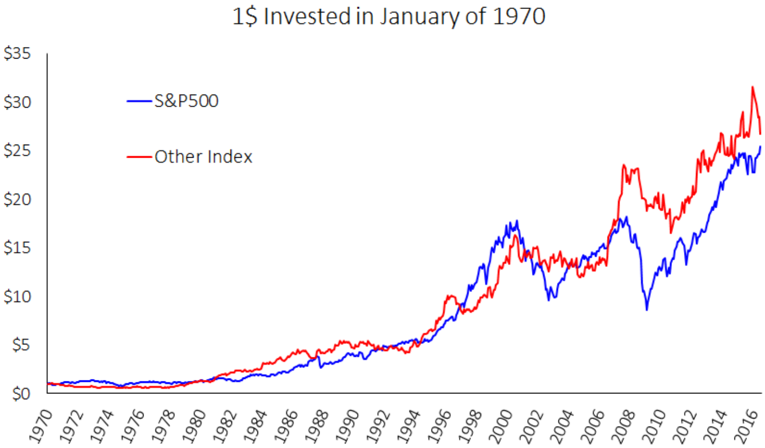
- n is the number of periods in a year. For instance, $n = 1$ for annual r_t and $n = 12$ for monthly r_t

Measuring Performance: Inflation Effect



$$1 + r_t^{real} = \frac{1 + r_t}{1 + i_t} \qquad \Leftrightarrow \qquad r_t^{real} \cong r_t - i_t$$

Returns as a Random Variable: Indices



Returns as a Random Variable: $\mathbb{E}[r_t]$ and $\sigma[r_t]$

Economic Scenario Next Year (s)	$p(s)$	$r(s)$	$r(s) - \mathbb{E}[r_t]$
Very High Growth	0.15	30%	20%
Growth $> \mathbb{E}[Growth]$	0.25	20%	10%
Growth $= \mathbb{E}[Growth]$	0.35	10%	0%
Growth $< \mathbb{E}[Growth]$	0.2	-5%	-15%
Recession	0.05	-40%	-50%

$$\mathbb{E}[r_t] = \sum_s p(s) \times r(s) = 10\%$$

$$\sigma[r_t] = \sqrt{\sum_s p(s) \times \{r(s) - \mathbb{E}[r_t]\}^2} = 16\%$$

Returns as a Random Variable: $\mathbb{E}[r_t]$ and $\sigma[r_t]$

Economic Scenario Next Year (s)	$p(s)$	$r(s)$	$r(s) - \mathbb{E}[r_t]$
Very High Growth	0.15	40%	30%
Growth $> \mathbb{E}[Growth]$	0.25	30%	20%
Growth $= \mathbb{E}[Growth]$	0.35	10%	0%
Growth $< \mathbb{E}[Growth]$	0.2	-20%	-30%
Recession	0.05	-60%	-70%

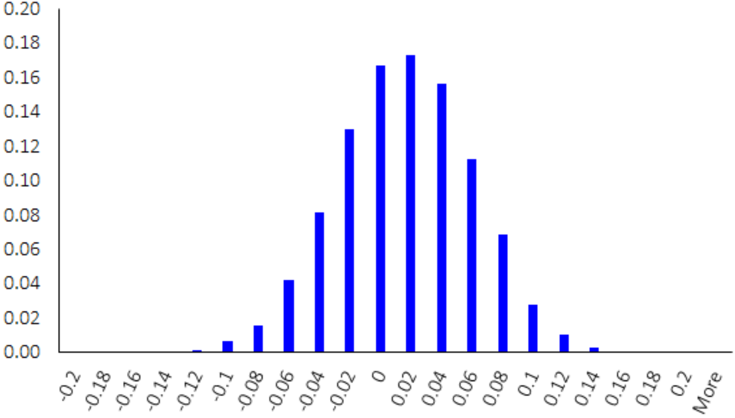
$$\mathbb{E}[r_t] = \sum_s p(s) \times r(s) = 10\%$$

$$\sigma[r_t] = \sqrt{\sum_s p(s) \times \{r(s) - \mathbb{E}[r_t]\}^2} = 26\%$$

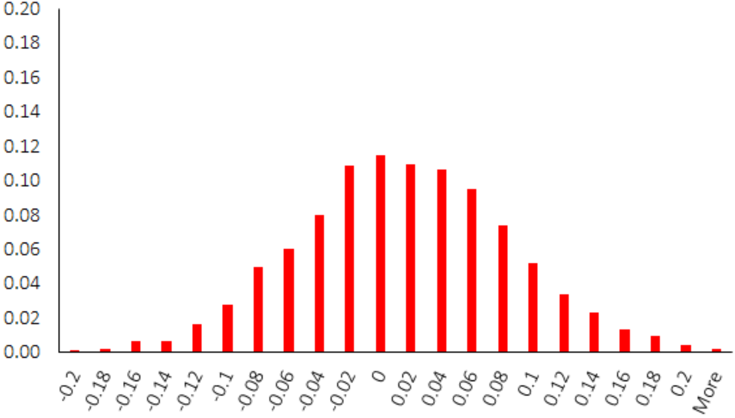
Modeling Returns: $r_t \sim \mathcal{N}(\mu, \sigma)$

- $\mu = \mathbb{E}[r_t]$ and $\sigma = \sigma[r_t]$
- Index simulation: I simulate r_t and use it to get future prices
- If r_t are truly normal, then only μ and σ matter (they describe entire distribution). In this case, the only measure of risk is σ
- If $r_{i,t}$ are normal, then so are portfolio returns: $r_{p,t} = w_1 \times r_{1,t} + w_2 \times r_{2,t} + ... + w_N \times r_{N,t}$
- If daily r_t are normal, annual r_t are not normal: horizon matters!

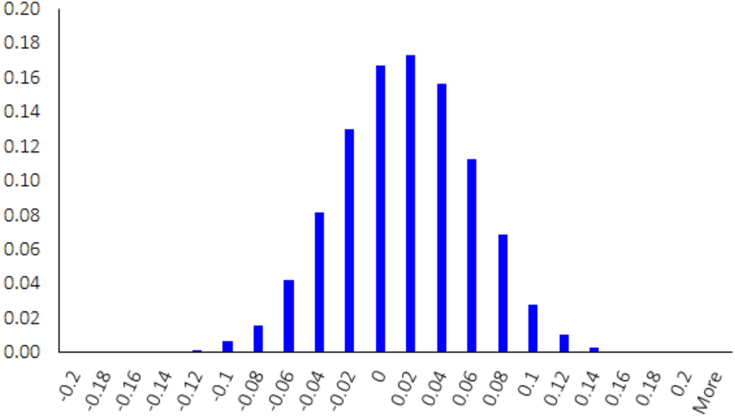
Modeling Returns: $r_t \sim \mathcal{N}(\mu = 0.6\%, \sigma = 4.4\%)$



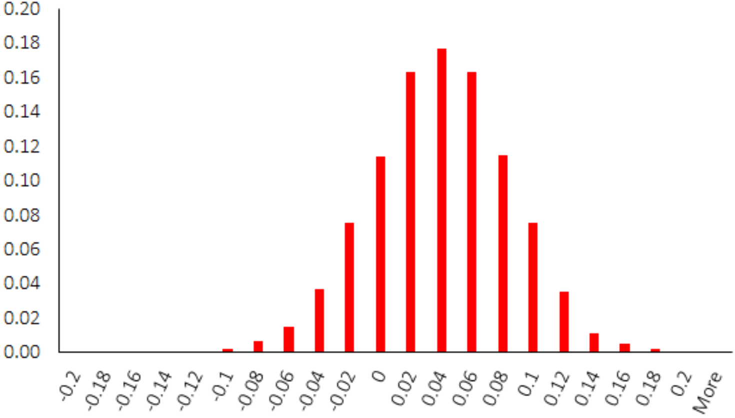
Modeling Returns: $r_t \sim \mathcal{N}(\mu = 0.6\%, \sigma = 7.0\%)$



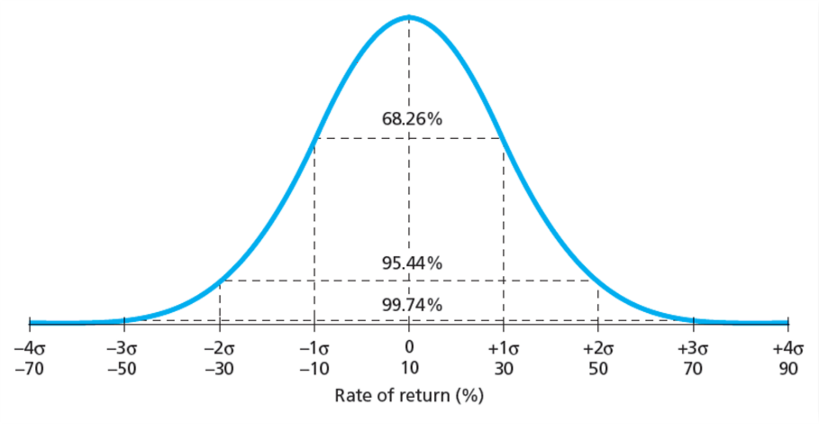
Modeling Returns: $r_t \sim \mathcal{N}(\mu = 0.6\%, \sigma = 4.4\%)$



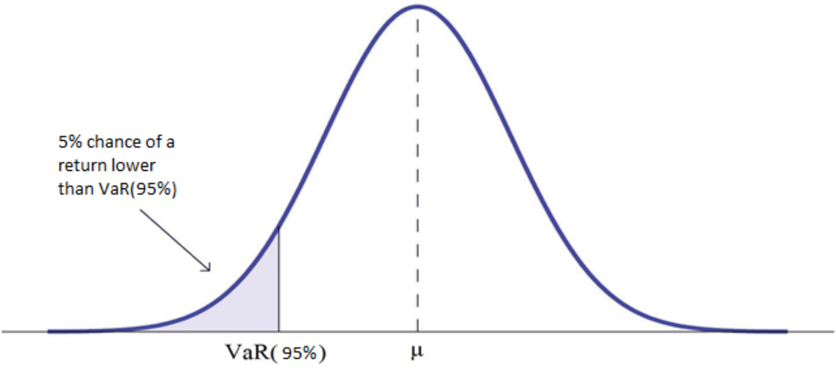
Modeling Returns: $r_t \sim \mathcal{N}\left(\mu = 3.0\%, \sigma = 4.4\%\right)$



Modeling Returns: $r_t \sim \mathcal{N}(\mu = 10\%, \sigma = 10\%)$

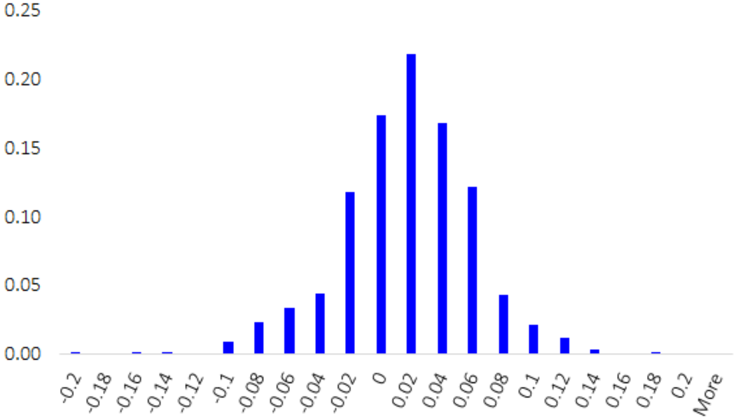


Modeling Returns: Tail Risk for Normal Returns?

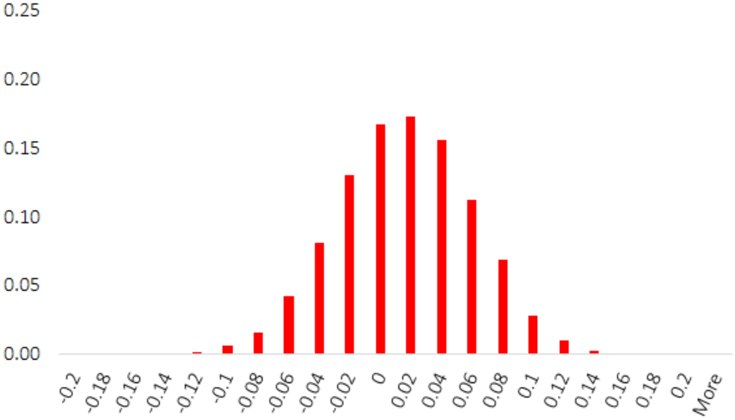


$$VaR\,(95\%) = \mathbb{E}\,[r_t] - 1.64 \times \sigma\,[r_t]$$

Modeling Returns: S&P 500



Modeling Returns: $r_t \sim \mathcal{N}\left(\mu_{S\&P}, \sigma_{S\&P}\right)$



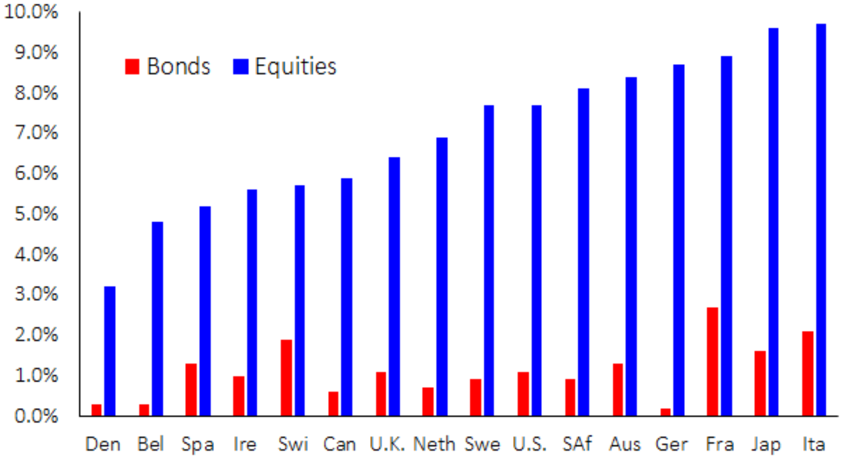
Data: Estimating Model from Time Series of r_t

- If $r_t \sim \mathcal{N}(\mu, \sigma)$, how can we estimate μ and σ from data?

- Recall:
$$\mu = \mathbb{E}[r_t] = \sum_s p(s) \times r(s)$$
$$\sigma = \sigma[r_t] = \sqrt{\sum_s p(s) \times \{r(s) - \mathbb{E}[r_t]\}^2}$$

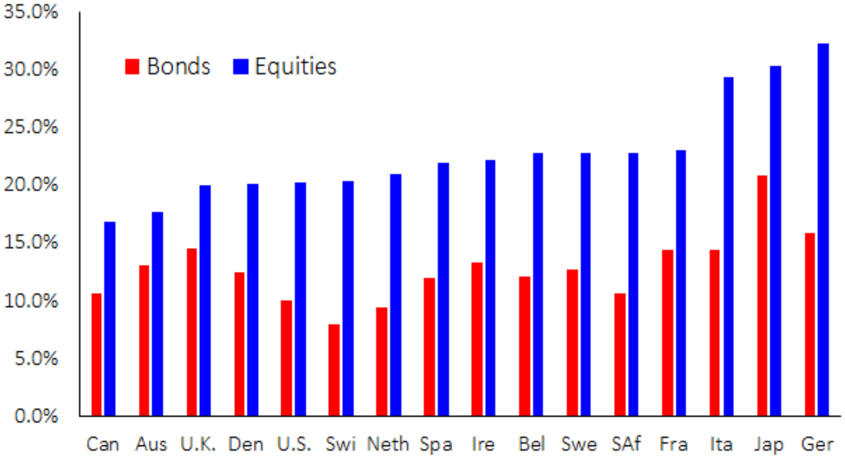
- Use each data observation as a "scenario" with equal probability:
$$\hat{\mu} = \frac{1}{T} \times \sum_{t=1}^{T} r_t = \overline{r}$$
$$\hat{\sigma} = \sqrt{\frac{1}{T-1} \times \sum_{t=1}^{T} \{r_t - \overline{r}\}^2}$$

Data: $(\bar{r} - \bar{r}_{TBill})$ around the World from 1900-2000



Source: Dimson et al (2002) - *Triumph of the optimists: 101 years of global investment returns*

Data: $\hat{\sigma}\left[r_t\right]$ around the World from 1900-2000



Source: Dimson et al (2002) - *Triumph of the optimists: 101 years of global investment returns*

If asset A daily r_t follow a normal distribution, then:

a) Asset A monthly r_t also do

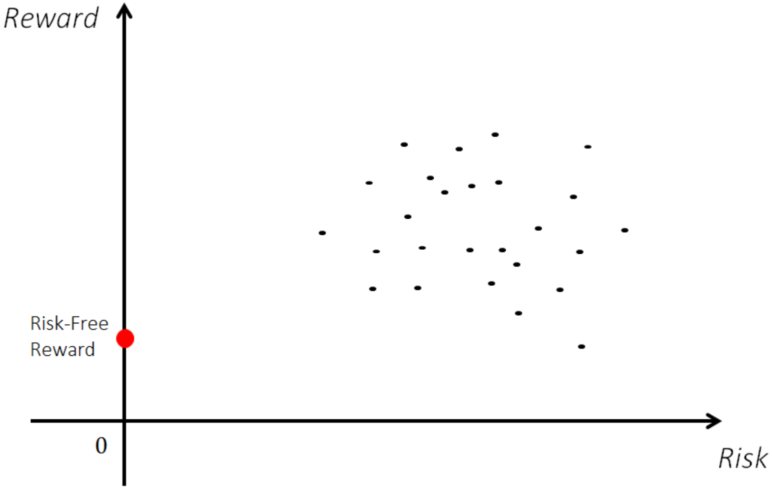
b) An increase in $\sigma[r_t]$ implies an increase in the tail risk the asset

c) Inflation does not matter for the real return on asset A

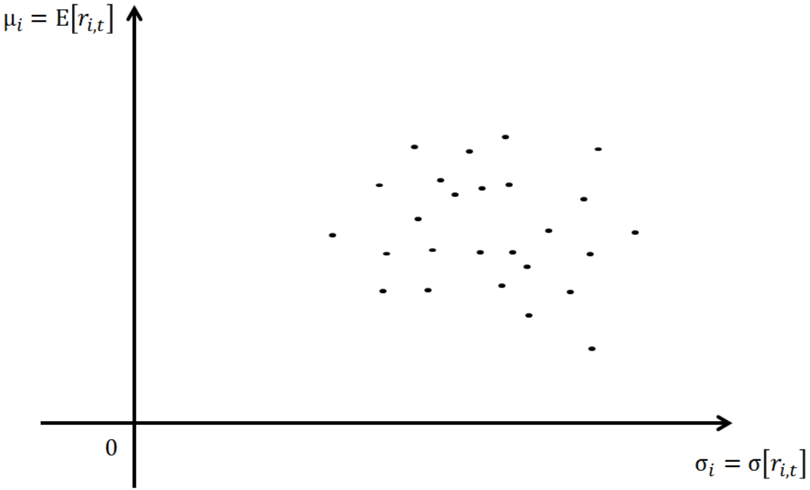
d) The geometric average return and the average return are the same

e) All we need to know about asset A in order to fully understand its daily returns distribution is $\sigma[r_t]$

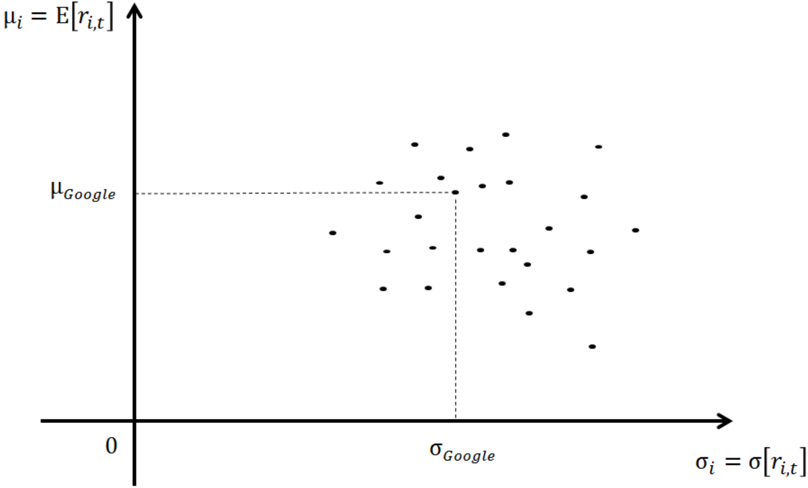
This Section: Find the “Best” Risky Portfolios



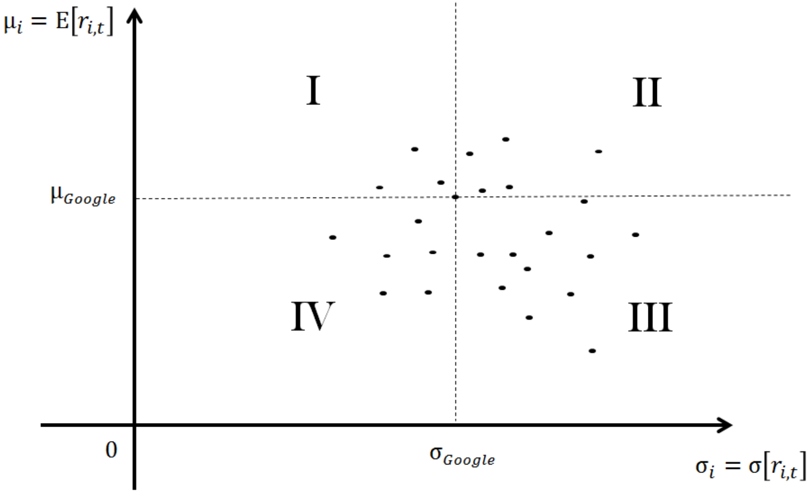
$\sigma[r_t] \times \mathbb{E}[r_t]$: Principle



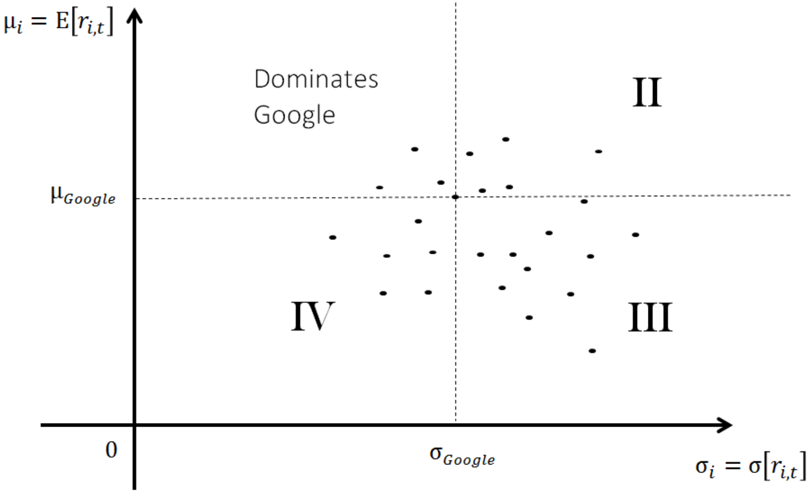
$\sigma[r_t] \times \mathbb{E}[r_t]$: Principle



$\sigma[r_t] \times \mathbb{E}[r_t]$: Principle



$\sigma[r_t] \times \mathbb{E}[r_t]$: Principle



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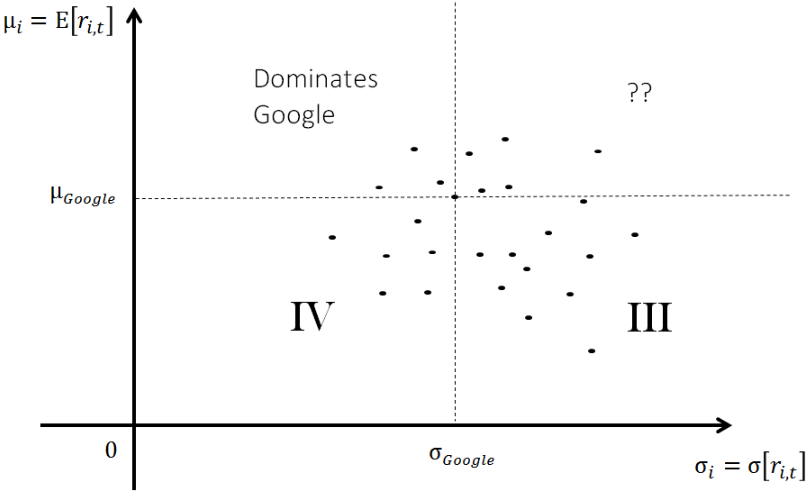
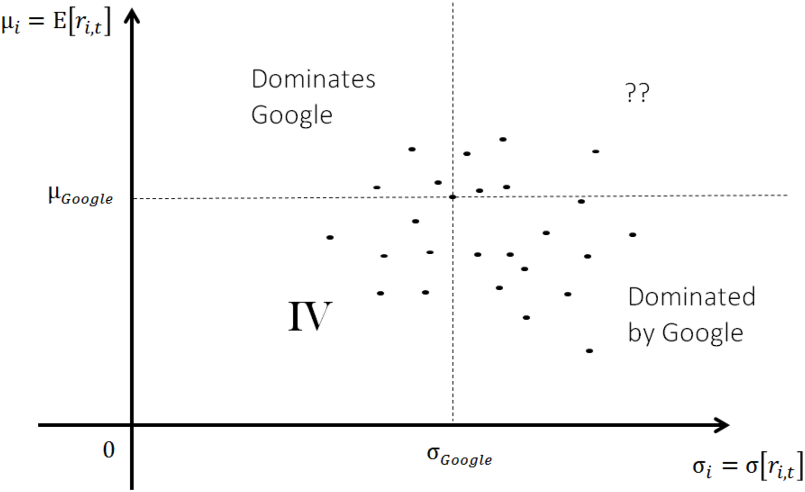
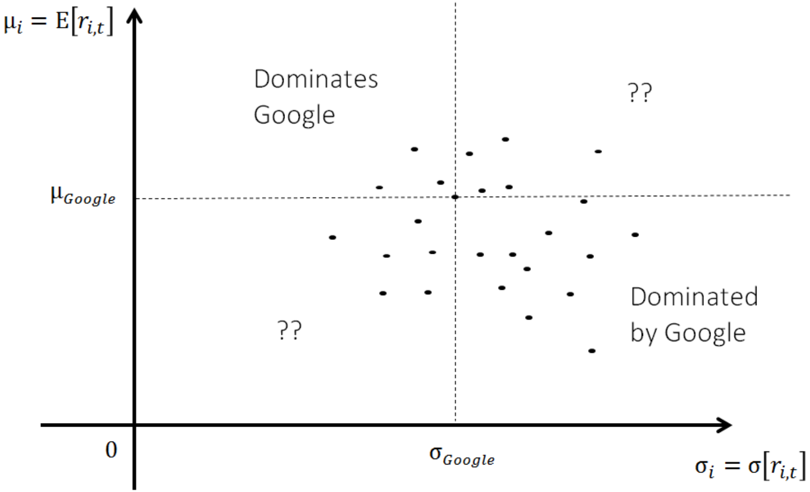


$\sigma[r_t] \times \mathbb{E}[r_t]$: Principle



$\sigma[r_t] \times \mathbb{E}[r_t]$: Principle



Diversification: Basic Principle

Diversification: Basic Principle

Diversification: Simple Example

Economic Scenario Next Year (s)	$p(s)$	r_{Equity}	r_{Gold}	$50\% - 50\%$ $r_{Portfolio}$
Very High Growth	0.15	30%	-12%	9%
Growth $> \mathbb{E}[Growth]$	0.25	20%	5%	13%
Growth $= \mathbb{E}[Growth]$	0.35	10%	15%	13%
Growth $< \mathbb{E}[Growth]$	0.2	-5%	20%	8%
Recession	0.05	-40%	25%	-8%
$\mathbb{E}[r_t] =$				
$\sigma[r_t] =$				

Diversification: Simple Example

Economic Scenario Next Year (s)	$p(s)$	r_{Equity}	r_{Gold}	50% – 50% $r_{Portfolio}$
Very High Growth	0.15	30%	-12%	9%
Growth $> \mathbb{E}[Growth]$	0.25	20%	5%	13%
Growth $= \mathbb{E}[Growth]$	0.35	10%	15%	13%
Growth $< \mathbb{E}[Growth]$	0.2	-5%	20%	8%
Recession	0.05	-40%	25%	-8%
$\mathbb{E}[r_t] =$		10%	10%	
$\sigma[r_t] =$		16%	11%	

Diversification: Simple Example

Economic Scenario Next Year (s)	$p(s)$	r_{Equity}	r_{Gold}	$50\% - 50\%$ $r_{Portfolio}$
Very High Growth	0.15	30%	-12%	9%
Growth $> \mathbb{E}[Growth]$	0.25	20%	5%	13%
Growth $= \mathbb{E}[Growth]$	0.35	10%	15%	13%
Growth $< \mathbb{E}[Growth]$	0.2	-5%	20%	8%
Recession	0.05	-40%	25%	-8%
$\mathbb{E}[r_t] =$		10%	10%	10%
$\sigma[r_t] =$		16%	11%	5%

Diversification: Simple Example

Economic Scenario Next Year (s)	$p(s)$	r_{Equity}	r_{Gold}	$40\% - 60\%$ $r_{Portfolio}$
Very High Growth	0.15	30%	-12%	5%
Growth $> \mathbb{E}[Growth]$	0.25	20%	5%	11%
Growth $= \mathbb{E}[Growth]$	0.35	10%	15%	13%
Growth $< \mathbb{E}[Growth]$	0.2	-5%	20%	10%
Recession	0.05	-40%	25%	-1%
$\mathbb{E}[r_t] =$		10%	10%	10%
$\sigma[r_t] =$		16%	11%	4%

Diversification: Simple Example

Economic Scenario Next Year (s)	$p(s)$	r_{Equity}	r_{Gold}	$50\% - 50\%$ $r_{Portfolio}$
Very High Growth	0.15	30%	-12%	9%
Growth $> \mathbb{E}[Growth]$	0.25	20%	5%	13%
Growth $= \mathbb{E}[Growth]$	0.35	10%	15%	13%
Growth $< \mathbb{E}[Growth]$	0.2	-5%	20%	8%
Recession	0.05	-40%	25%	-8%
$\mathbb{E}[r_t] =$		10%	10%	10%
$\sigma[r_t] =$		16%	11%	5%

Diversification: Simple Example

Economic Scenario Next Year (s)	$p(s)$	r_{Equity}	r_{Gold}	$50\% - 50\%$ $r_{Portfolio}$
Very High Growth	0.15	30%	-7%	12%
Growth $> \mathbb{E}[Growth]$	0.25	20%	5%	13%
Growth $= \mathbb{E}[Growth]$	0.35	10%	15%	13%
Growth $< \mathbb{E}[Growth]$	0.2	-5%	26%	11%
Recession	0.05	-40%	-10%	-25%
$\mathbb{E}[r_t] =$		10%	10%	10%
$\sigma[r_t] =$		16%	11%	8%

Diversification: Covariance & Correlation

Economic Scenario Next Year (s)	$p(s)$	r_{Equity}	r_{Gold}	$(r_E - \mathbb{E}[r_E]) \times (r_G - \mathbb{E}[r_G])$
Very High Growth	0.15	30%	-7%	-0.034
Growth $> \mathbb{E}[Growth]$	0.25	20%	5%	-0.005
Growth $= \mathbb{E}[Growth]$	0.35	10%	15%	0.000
Growth $< \mathbb{E}[Growth]$	0.2	-5%	26%	-0.024
Recession	0.05	-40%	-10%	0.100
$\mathbb{E}[r_t] =$		10%	10%	
$\sigma[r_t] =$		16%	11%	

$$Cov[r_E, r_G] = \sum_s p(s) \times \{r_E(s) - \mathbb{E}[r_E]\} \times \{r_G(s) - \mathbb{E}[r_G]\}$$

$$\rho[r_E, r_G] = \frac{Cov[r_E, r_G]}{\sigma[r_E] \times \sigma[r_G]} = -0.34$$

Diversification: Two Risky Assets

- Consider the portfolio $r_p = w_A \cdot r_A + w_B \cdot r_B$
- Portfolio expected return, $\mathbb{E}[r_p]$, is given by:

$$\mathbb{E}[r_p] = w_A \cdot \mathbb{E}[r_A] + w_B \cdot \mathbb{E}[r_B]$$

- Portfolio variance, $\sigma^2[r_p] = \sigma_p^2$, is given by:

$$\sigma_p^2 = w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2 \cdot w_A \cdot w_B \cdot Cov[r_A, r_B]$$

$$= w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2 \cdot w_A \cdot w_B \cdot \sigma_A \cdot \sigma_B \cdot \rho[r_A, r_B]$$

Diversification: Two Risky Assets

- When do we have diversification: $\sigma_p < w_A \cdot \sigma_A + w_B \cdot \sigma_B$?

- Suppose $\rho\left[r_A, r_B\right] = 1$, then:

$$\sigma_p^2 = w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2 \cdot w_A \cdot w_B \cdot \sigma_A \cdot \sigma_B \cdot 1$$

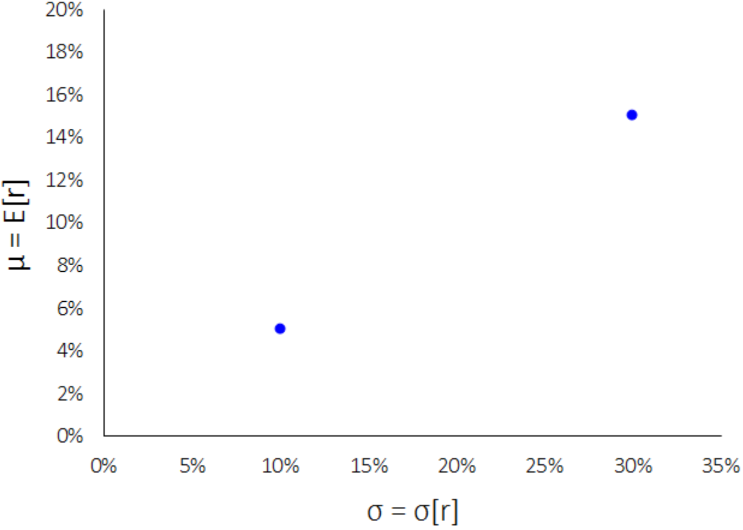
$$= \left(w_A \cdot \sigma_A + w_B \cdot \sigma_B\right)^2$$

$$\Downarrow$$

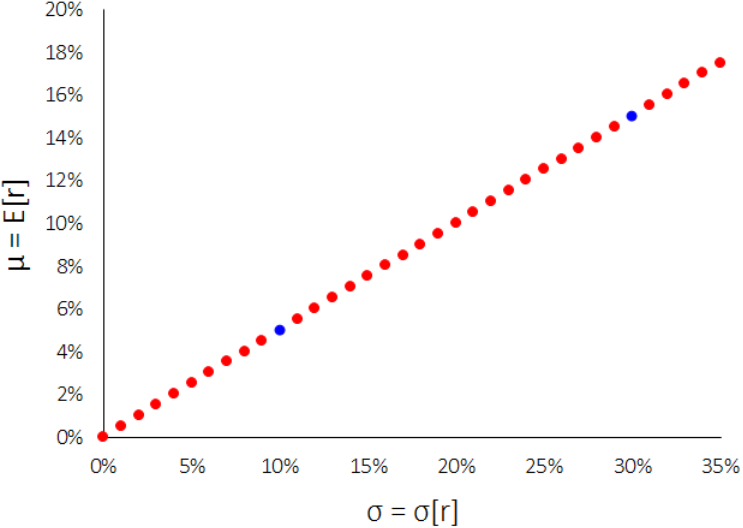
$$\sigma_p = w_A \cdot \sigma_A + w_B \cdot \sigma_B$$

- Any $\rho\left[r_A, r_B\right] < 1$ produces $\sigma_p < w_A \cdot \sigma_A + w_B \cdot \sigma_B$

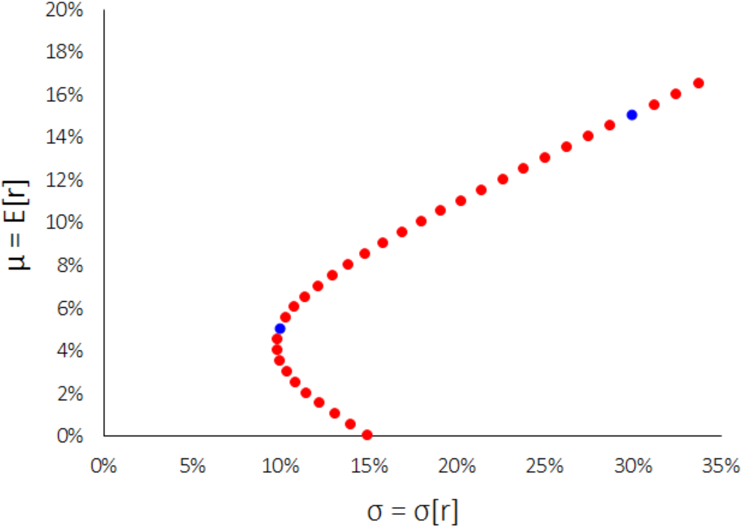
Diversification: Two Risky Assets



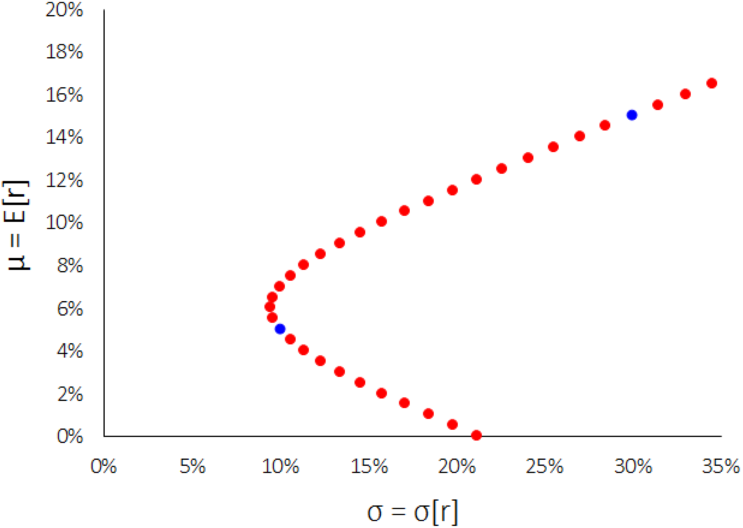
Diversification: Two Risky Assets ($\rho = 1.0$)



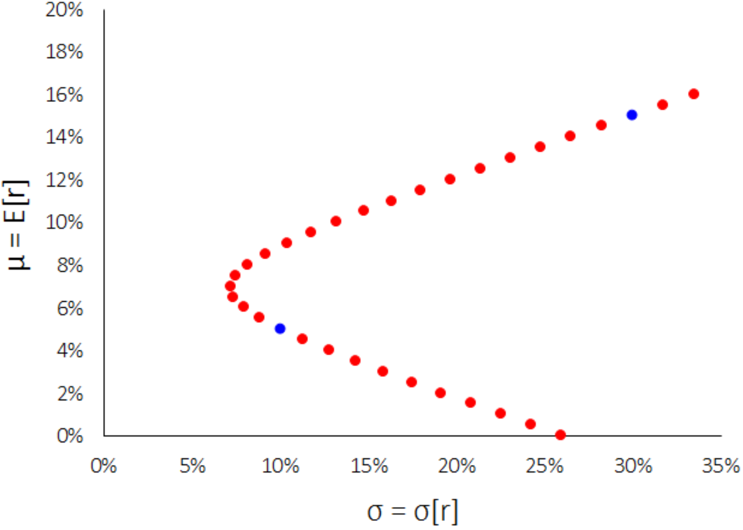
Diversification: Two Risky Assets ($\rho = 0.5$)



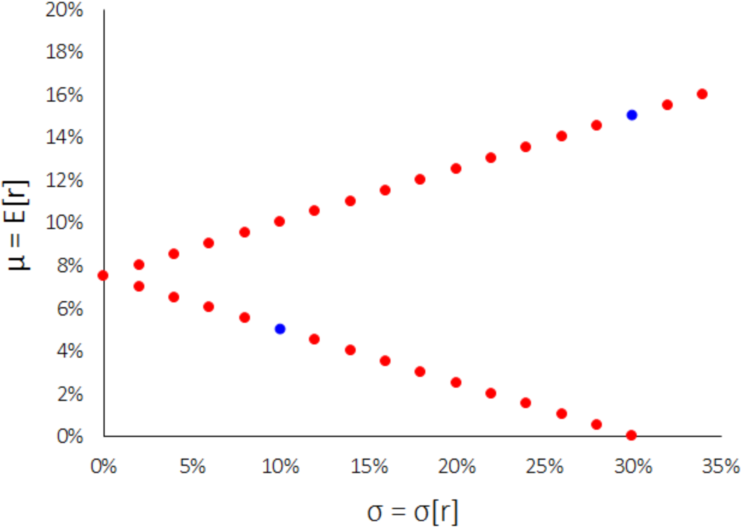
Diversification: Two Risky Assets ($\rho = 0.0$)



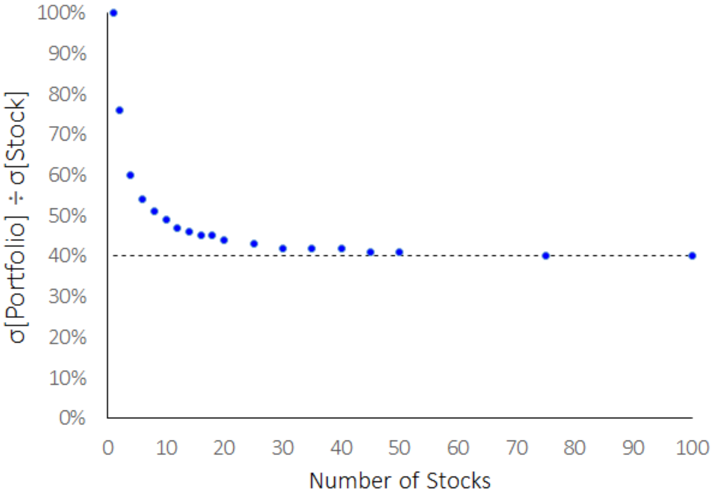
Diversification: Two Risky Assets ($\rho = -0.5$)



Diversification: Two Risky Assets ($\rho = -1.0$)

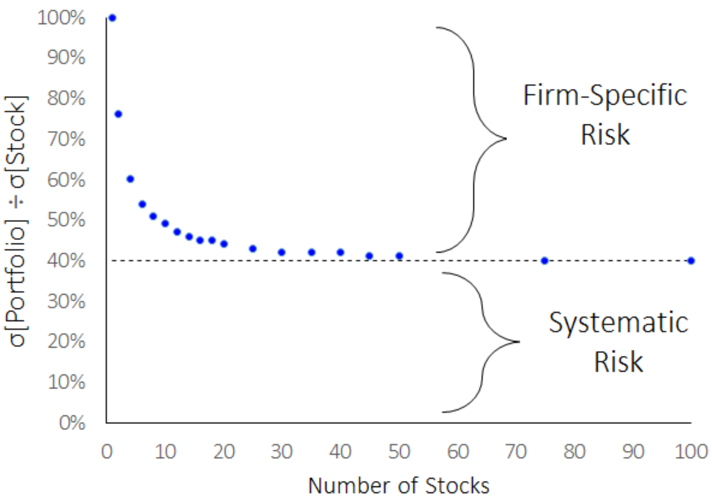


Diversification: Systematic × Firm-Specific Risk



Source: "Statman (1987) - *How many stocks make a diversified portfolio*"

Diversification: Systematic × Firm-Specific Risk



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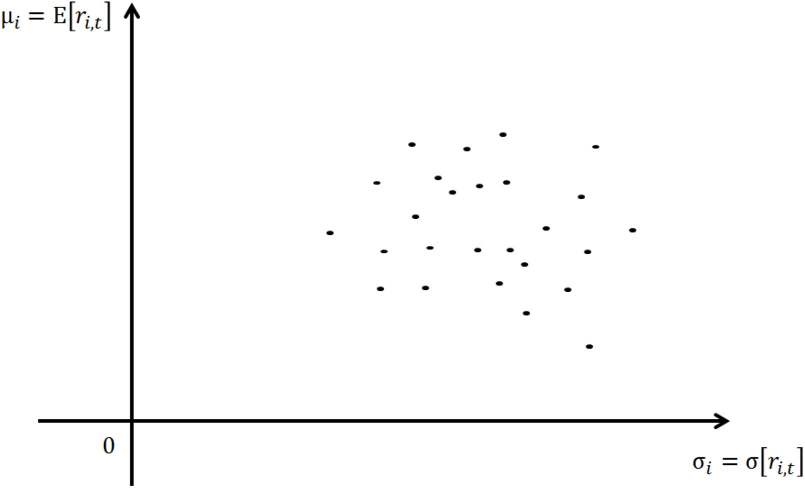
$\sigma[r_t] \times \mathbb{E}[r_t]$: Efficient Frontier



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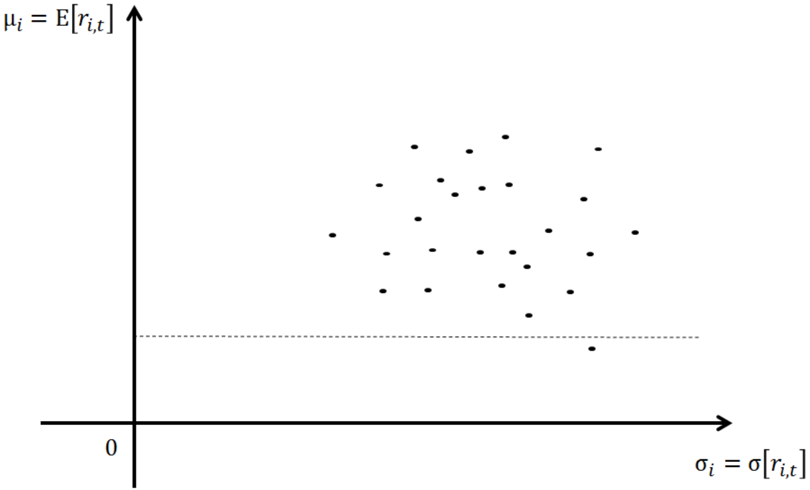

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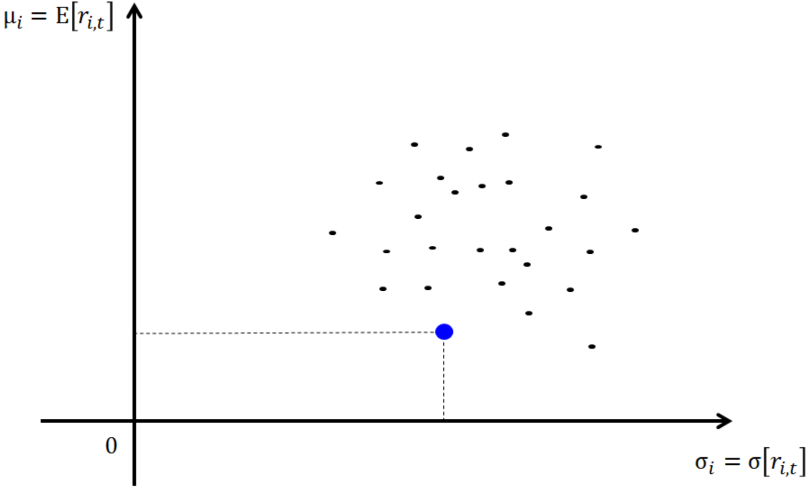
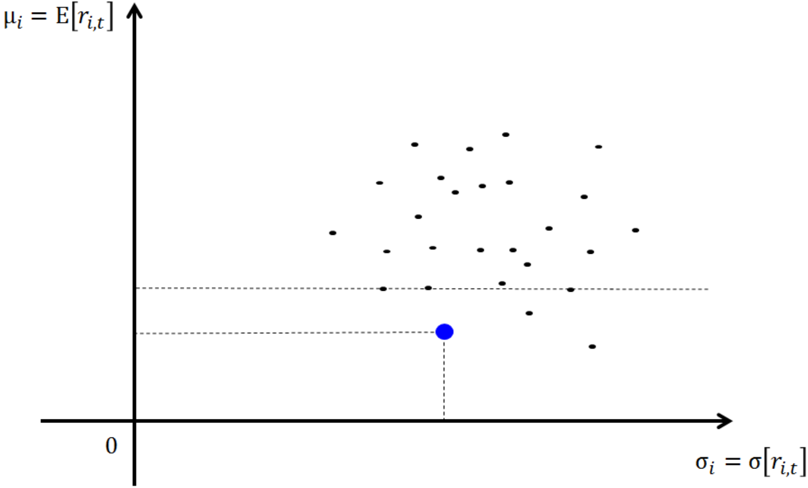
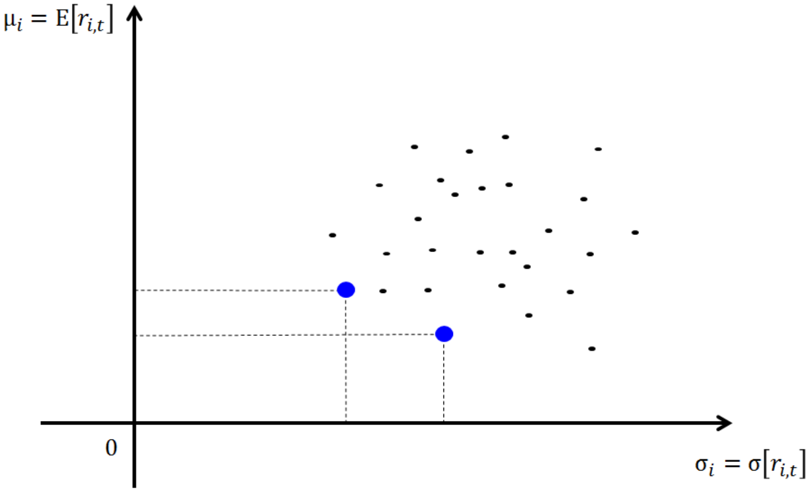


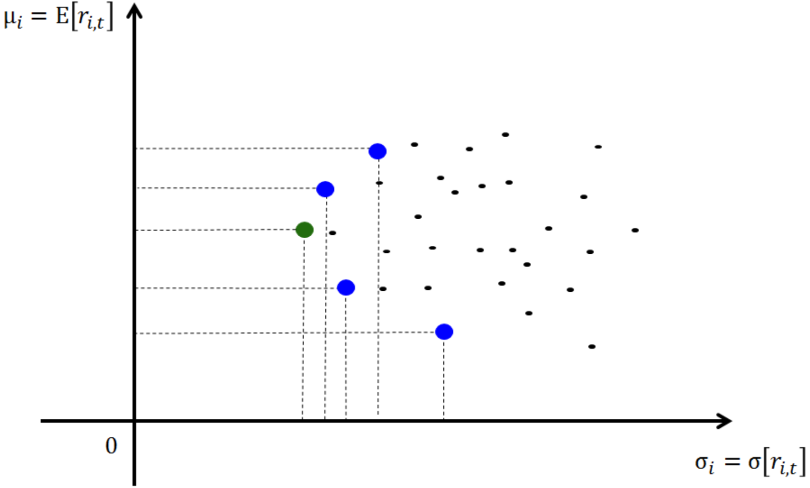
$\sigma[r_t] \times \mathbb{E}[r_t]$: Efficient Frontier



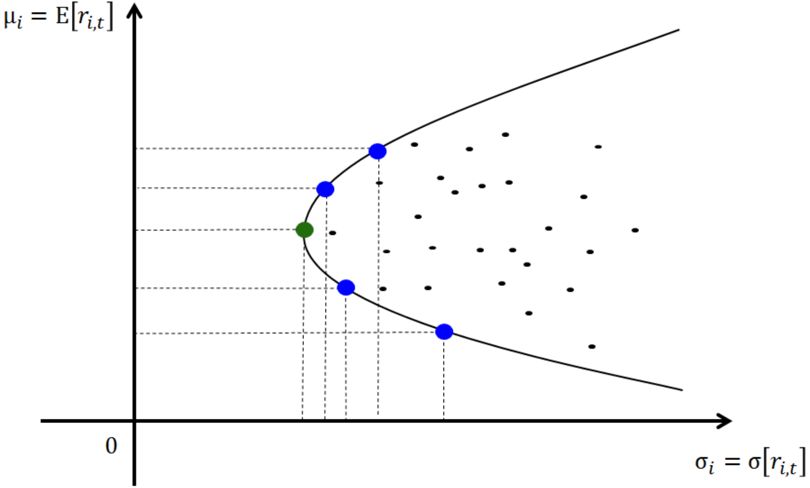
$\sigma[r_t] \times \mathbb{E}[r_t]$: Efficient Frontier



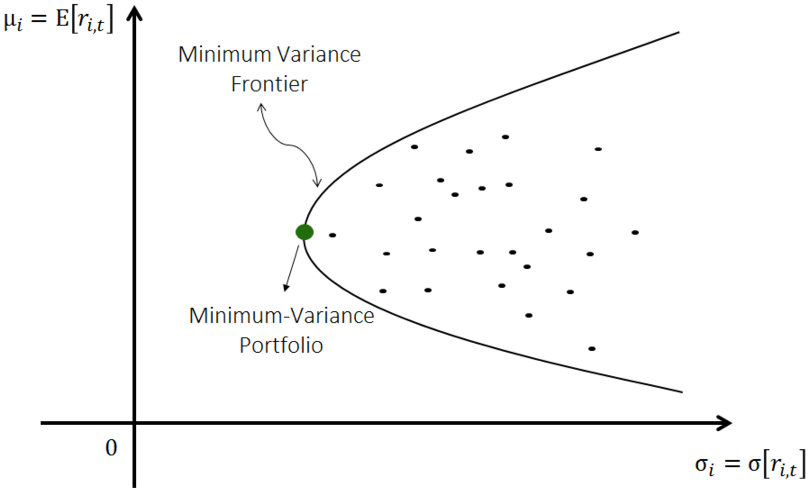
$\sigma[r_t] \times \mathbb{E}[r_t]$: Efficient Frontier



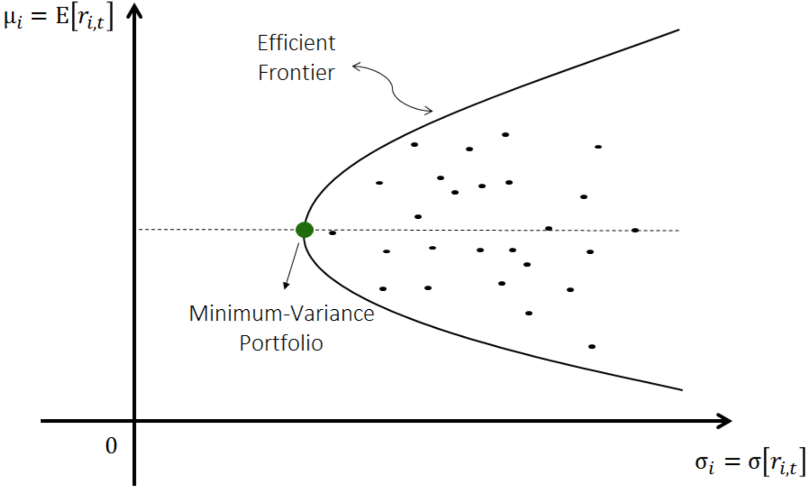
$\sigma[r_t] \times \mathbb{E}[r_t]$: Efficient Frontier



$\sigma[r_t] \times \mathbb{E}[r_t]$: Efficient Frontier



$\sigma[r_t] \times \mathbb{E}[r_t]$: Efficient Frontier



Which of the following is false regarding the Efficient Frontier?

a) It relies on estimates of risk, $\sigma[r_t]$, reward, $\mathbb{E}[r_t]$, and covariances for multiple assets

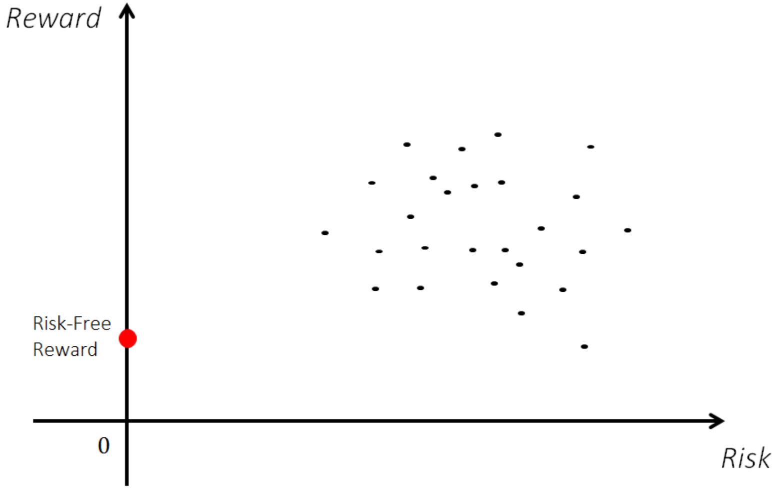
b) Any portfolio formed by first selecting a target $\mathbb{E}[r_t]$ and then choosing the portfolio with minimum risk among the ones with such target $\mathbb{E}[r_t]$ is an efficient portfolio

c) It restricts the set of potential portfolios an investor should choose from if he measures risk by $\sigma[r_t]$ and reward by $\mathbb{E}[r_t]$

d) It depends heavily on the power of diversification

e) With only two assets, the more negatively correlated they are the better is the efficient frontier investors can form using them

This Section: Adding Risk Free Asset



Combining Risky Asset with Risk-Free Asset

- Recall that for $r_p = w_A \cdot r_A + w_B \cdot r_B$ we have:

$$\mathbb{E}[r_p] = w_A \cdot \mathbb{E}[r_A] + w_B \cdot \mathbb{E}[r_B]$$

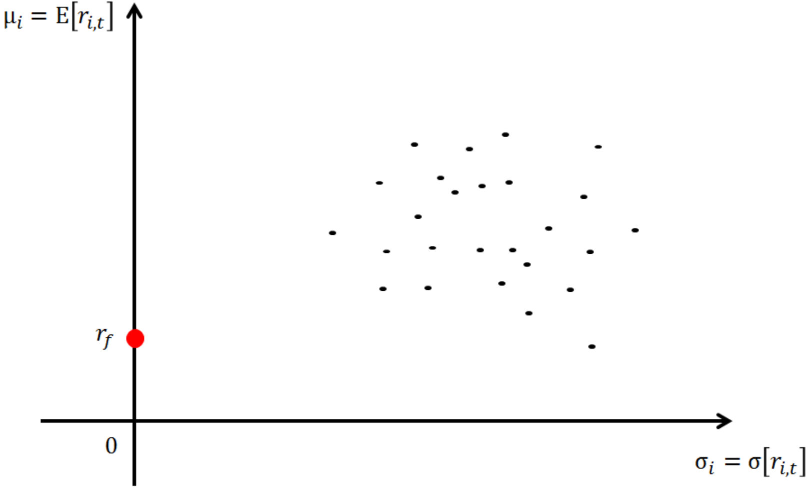
$$\sigma_p^2 = w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2 \cdot w_A \cdot w_B \cdot \sigma_A \cdot \sigma_B \cdot \rho[r_A, r_B]$$

- This means that $r_p = w_A \cdot r_A + w_f \cdot r_f$ satisfies:

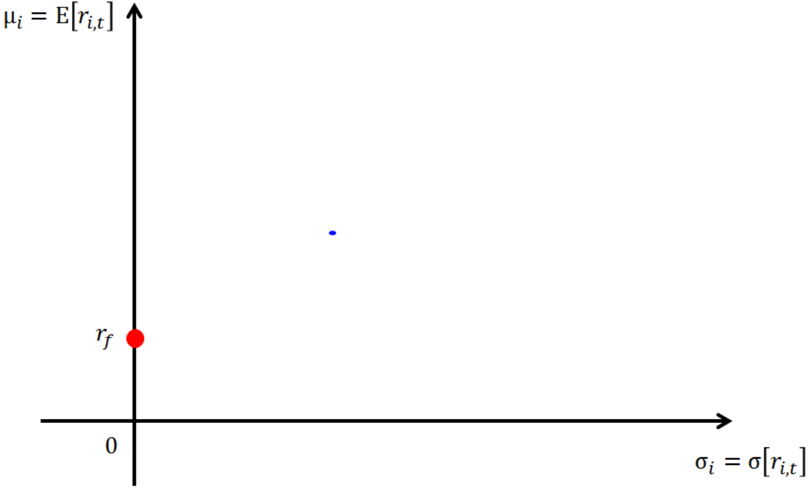
$$\mathbb{E}[r_p] = w_A \cdot \mathbb{E}[r_A] + w_f \cdot r_f$$

$$\sigma_p^2 = w_A^2 \cdot \sigma_A^2 \implies \sigma_p = w_A \cdot \sigma_A + w_f \cdot 0$$

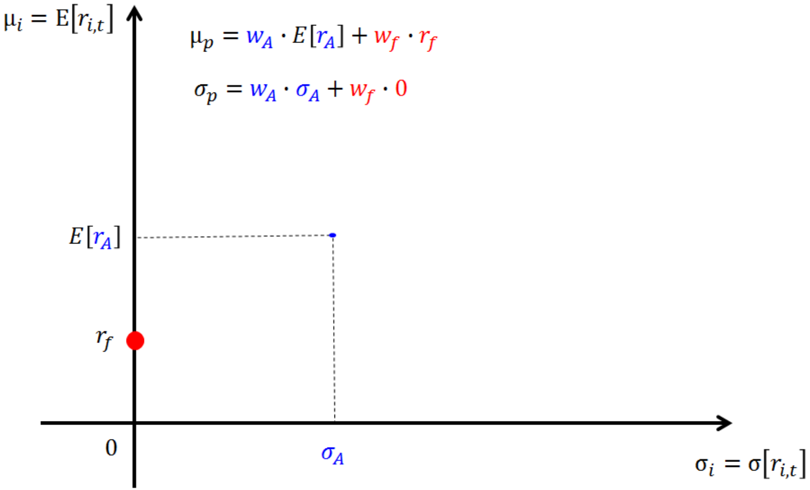
Combining Risky Asset with Risk-Free Asset



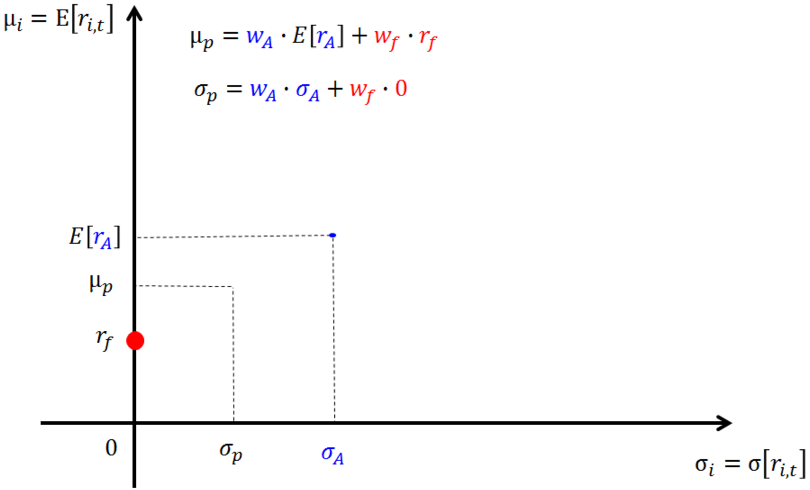
Combining Risky Asset with Risk-Free Asset



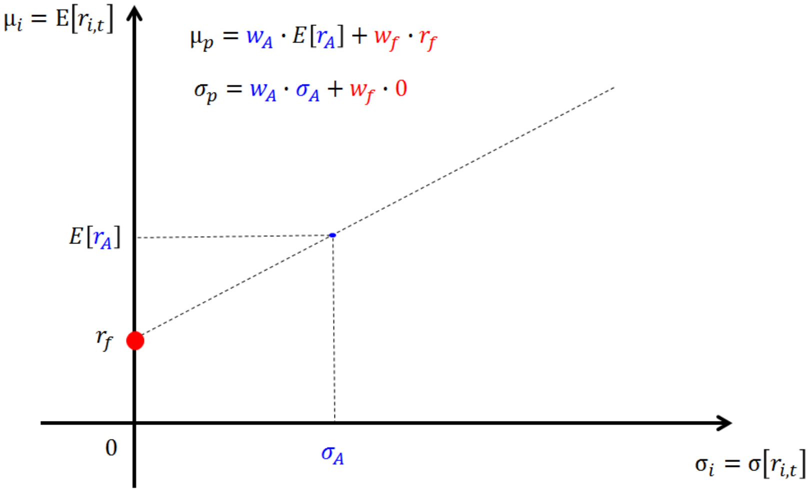
Combining Risky Asset with Risk-Free Asset



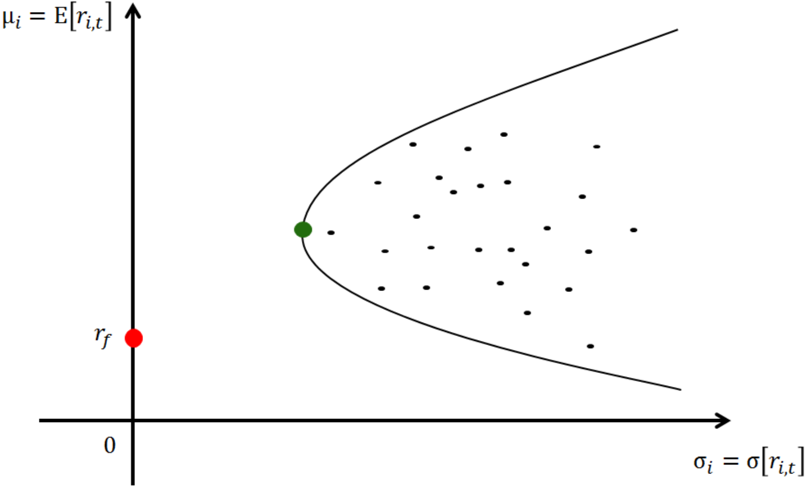
Combining Risky Asset with Risk-Free Asset



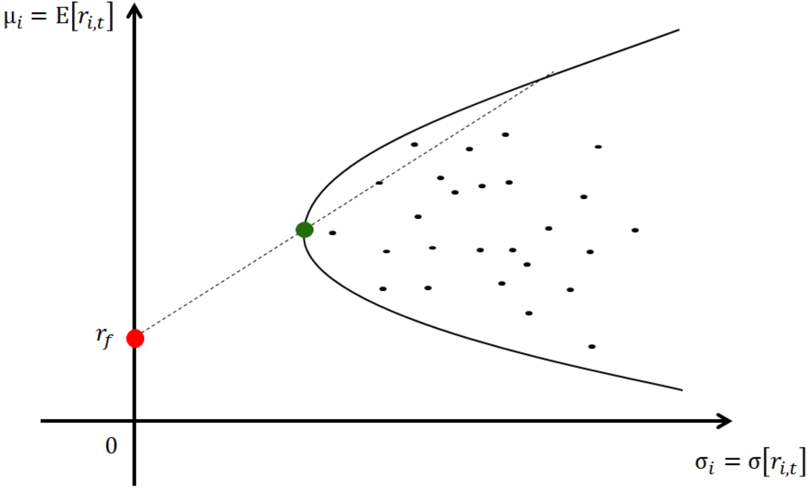
Combining Risky Asset with Risk-Free Asset



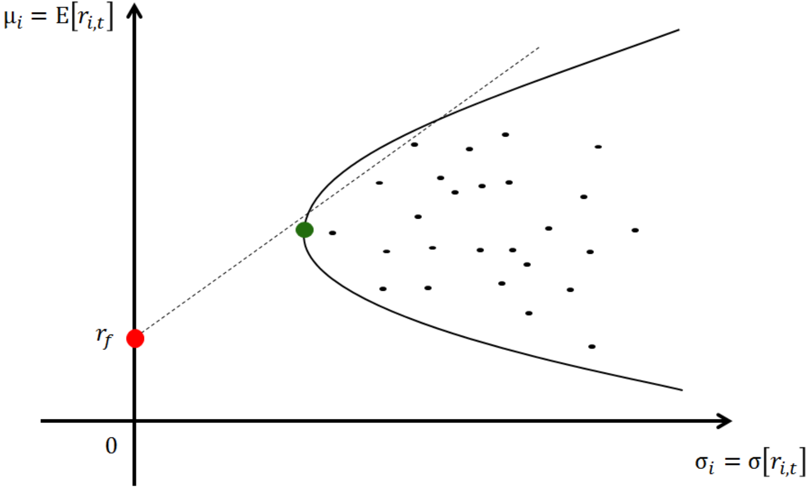
$\sigma[r_t] \times \mathbb{E}[r_t]$: Capital Allocation Line



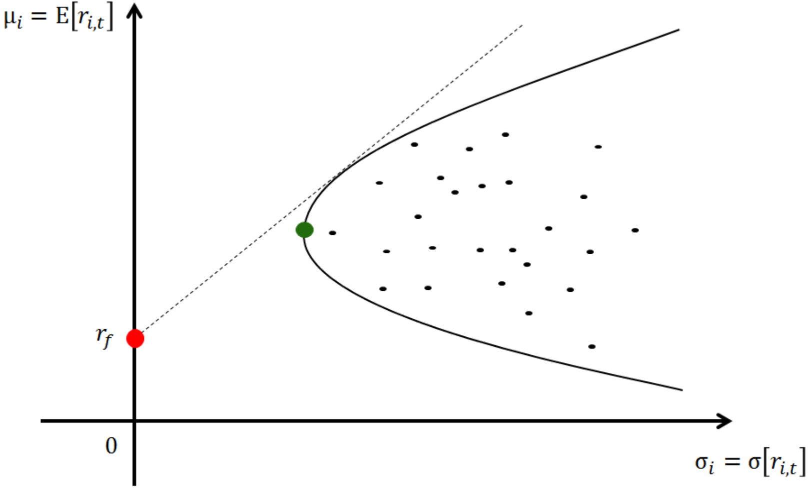
$\sigma[r_t] \times \mathbb{E}[r_t]$: Capital Allocation Line



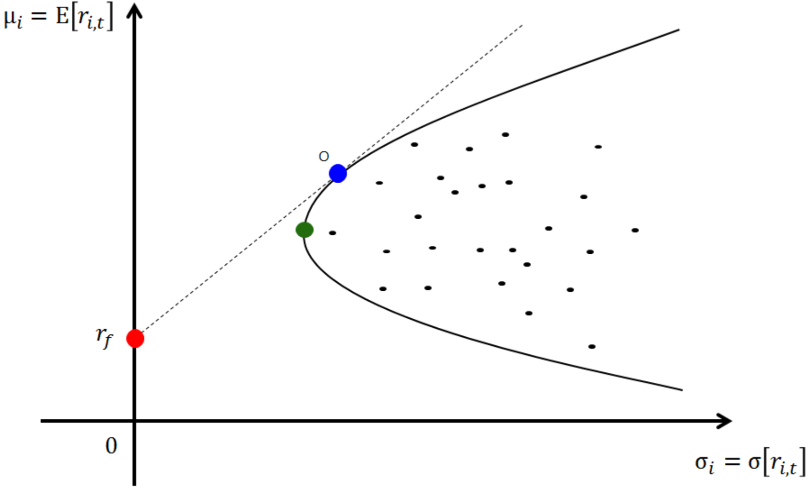
$\sigma[r_t] \times \mathbb{E}[r_t]$: Capital Allocation Line



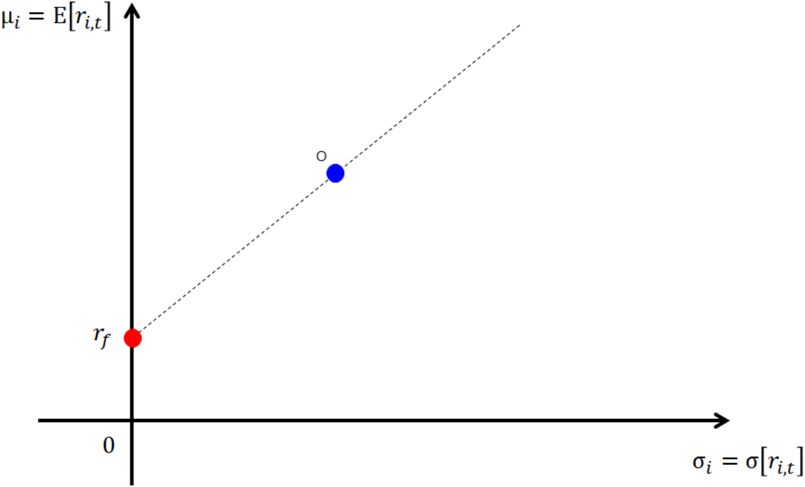
$\sigma[r_t] \times \mathbb{E}[r_t]$: Capital Allocation Line



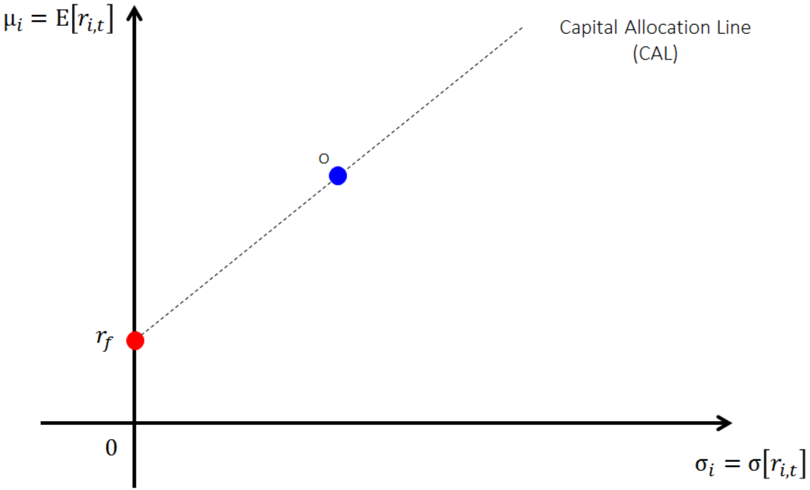
$\sigma[r_t] \times \mathbb{E}[r_t]$: Capital Allocation Line



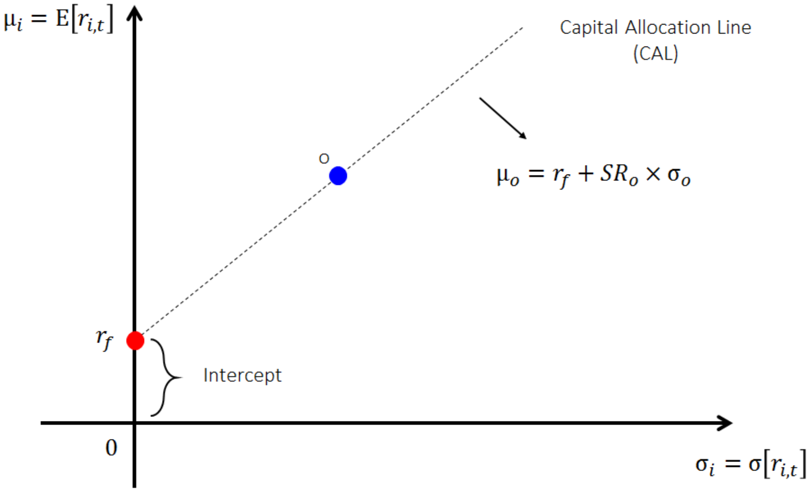
$\sigma[r_t] \times \mathbb{E}[r_t]$: Capital Allocation Line



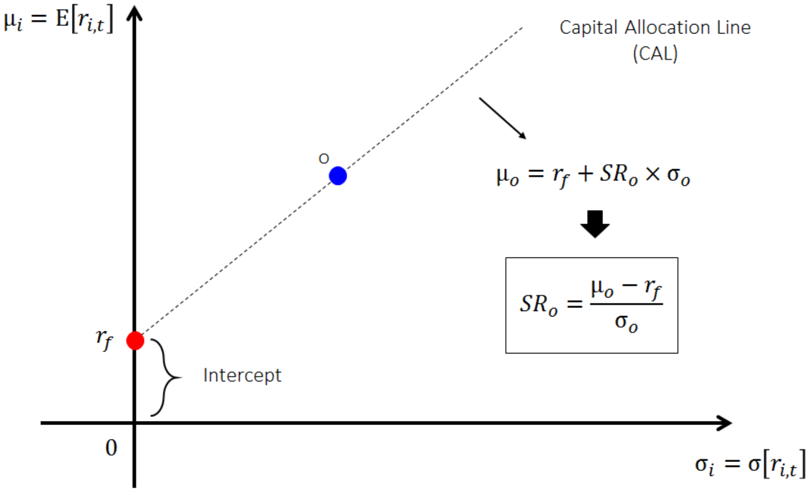
$\sigma[r_t] \times \mathbb{E}[r_t]$: Capital Allocation Line



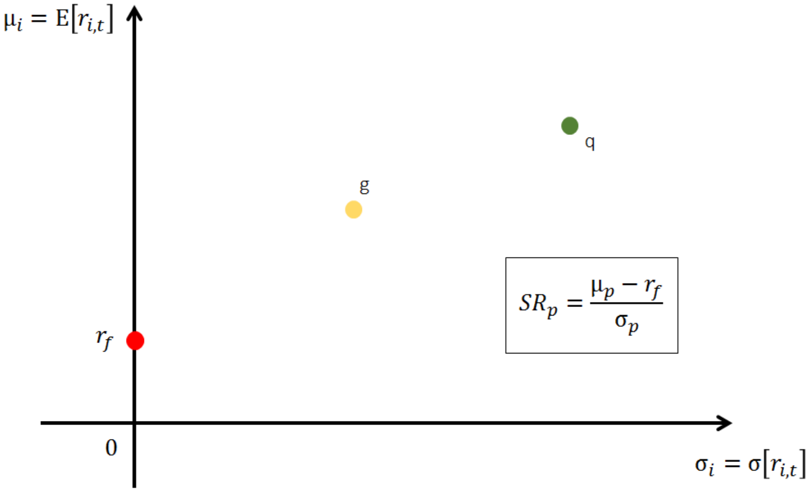
$\sigma[r_t] \times \mathbb{E}[r_t]$: Sharpe Ratio



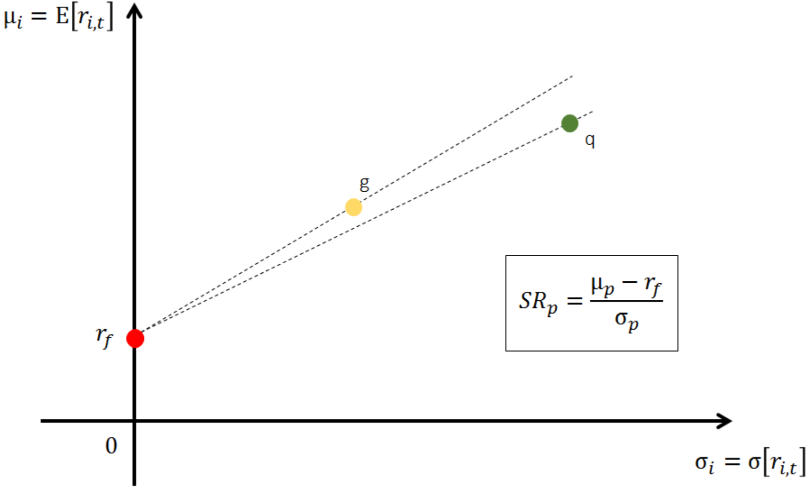
$\sigma[r_t] \times \mathbb{E}[r_t]$: Sharpe Ratio



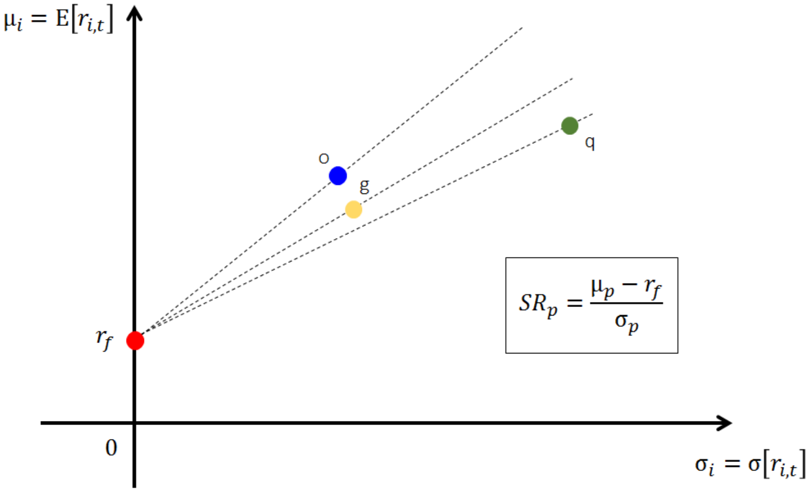
$\sigma[r_t] \times \mathbb{E}[r_t]$: Portfolio Comparison using Sharpe Ratio



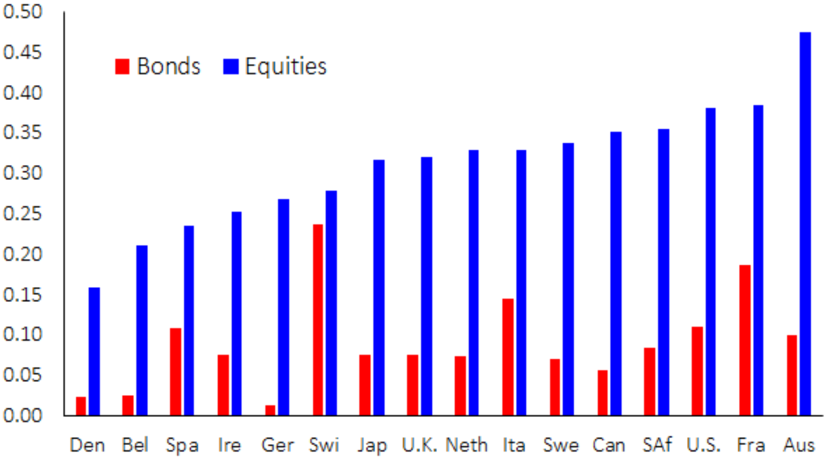
$\sigma[r_t] \times \mathbb{E}[r_t]$: Portfolio Comparison using Sharpe Ratio



$\sigma[r_t] \times \mathbb{E}[r_t]$: Portfolio Comparison using Sharpe Ratio

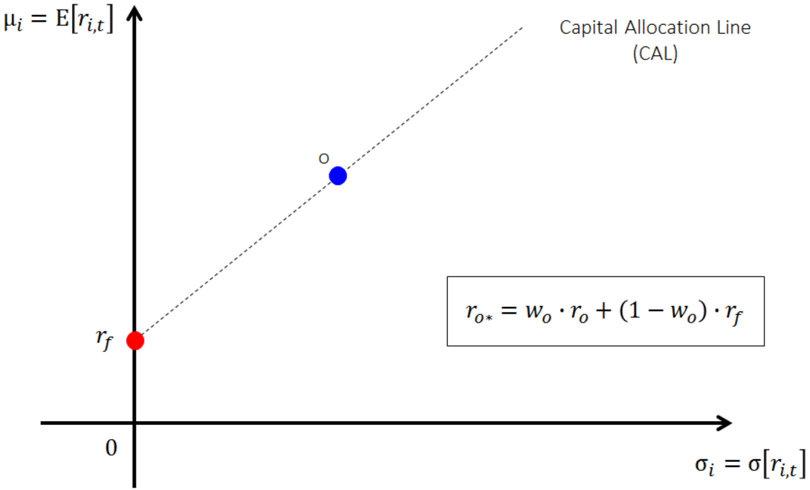


Sharpe Ratios around the World from 1900-2000

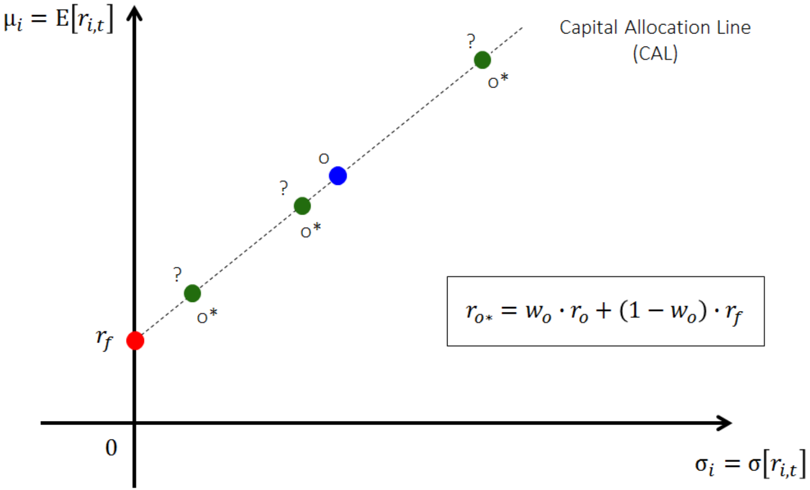


Source: Dimson et al (2002) - *Triumph of the optimists: 101 years of global investment returns*

$\sigma[r_t] \times \mathbb{E}[r_t]$: Complete Portfolio o^*



$\sigma[r_t] \times \mathbb{E}[r_t]$: Complete Portfolio o^*



Properties of $\mathbb{E}[r_{o^*}]$ and $\sigma[r_{o^*}]$

- Recall that for $r_p = w_A \cdot r_A + w_B \cdot r_B$ we have:

$$\mathbb{E}[r_p] = w_A \cdot \mathbb{E}[r_A] + w_B \cdot \mathbb{E}[r_B]$$

$$\sigma_p^2 = w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2 \cdot w_A \cdot w_B \cdot \sigma_A \cdot \sigma_B \cdot \rho[r_A, r_B]$$

- This means that $r_{o*} = w_o \cdot r_o + (1 - w_o) \cdot r_f$ satisfies:

$$\mathbb{E}[r_{o*}] = w_o \cdot \mathbb{E}[r_o] + (1 - w_o) \cdot r_f$$

$$\sigma_{o^*}^2 = w_o^2 \cdot \sigma_o^2 \quad \Longrightarrow \quad \sigma_{o^*} = w_o \cdot \sigma_o$$

Deciding on Complete Portfolio o^*

- Investors happiness depends on $\sigma_{o^*} = \sigma\left[r_{o^*}\right]$ and $\mu_{o^*} = \mathbb{E}\left[r_{o^*}\right]$

- We can model investors happiness (called utility function) as:

$$\begin{aligned} U\left(\sigma_{o^*}, \mu_{o^*}\right) &= \mu_{o^*} - 0.5 \cdot A \cdot \sigma_{o^*}^2 \\ &= \underbrace{w_o \cdot \mu_o + \left(1 - w_o\right) r_f}_{\mu_{o^*}} - 0.5 \cdot A \cdot \underbrace{w_o^2 \cdot \sigma_o^2}_{\sigma_{o^*}^2} \end{aligned}$$

- If investors select w_o to maximize their happiness we have:

$$\begin{aligned} w_o &= \frac{1}{A \cdot \sigma_o^2} \cdot \left(\mu_o - r_f\right) \\ &= \frac{1}{A \cdot \sigma_o} SR_o \quad \Longrightarrow \quad SR_o = w_o \cdot A \cdot \sigma_o \end{aligned}$$

The task of forming the "complete portfolio" (called o^*), is separated into two different tasks: (i) finding the "optimal" risky portfolio (called o) and (ii) getting o^* by combining portfolio o with r_f. Which of the following is true regarding this process:

a) Two investors using the same inputs to step (i) always end up with the same risky portfolio, o

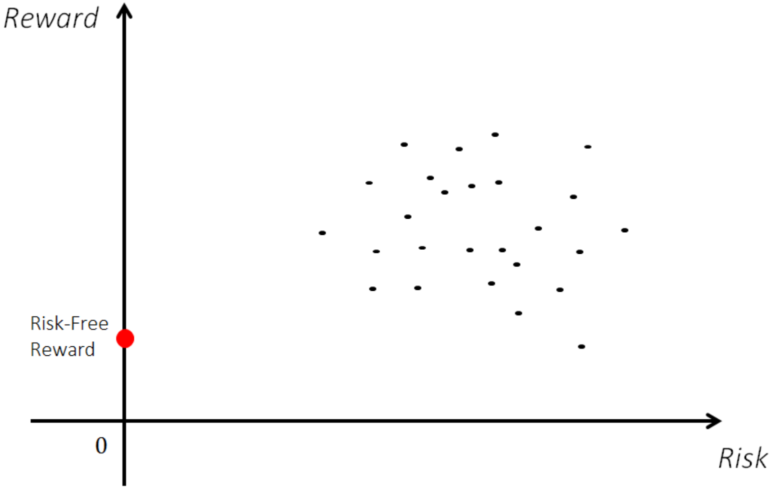
b) Step (ii) is investor specific. However, if two investors have the same risk aversion, they always end up with the same o^*

c) Portfolio o can only be found if we first find the entire efficient frontier

d) The negative covariance between r_f and o is very important to decide on portfolio o^*

e) Portfolio o is the portfolio with lowest risk among all possible portfolios that exclude the risk-free rate

This Section: Inputs to $\sigma[r_t] \times \mathbb{E}[r_t]$ Framework



Estimating Covariances

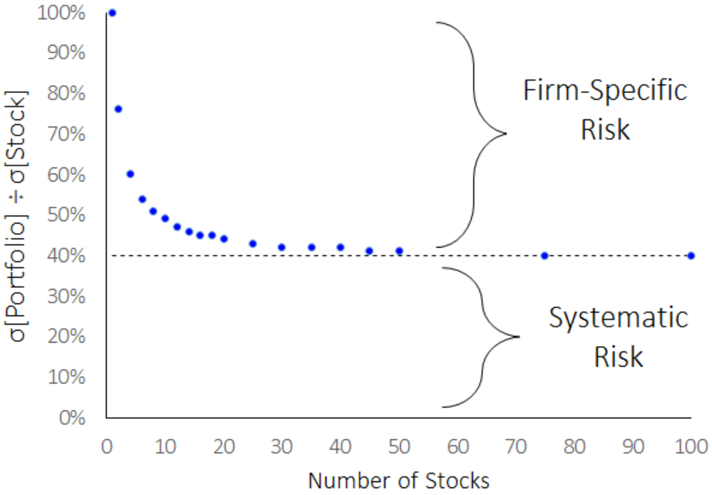
- Use each data observation as a "scenario" with equal probability:

$$Cov\left[r_A, r_B\right] = \sum_s p\left(s\right) \times \left\{r_A\left(s\right) - \mathbb{E}\left[r_A\right]\right\} \times \left\{r_B\left(s\right) - \mathbb{E}\left[r_B\right]\right\}$$

$$\widehat{Cov}\left[r_A, r_B\right] = \frac{1}{T-1} \times \sum_{t=1}^{T} \left\{r_{A,t} - \bar{r}_A\right\} \times \left\{r_{B,t} - \bar{r}_B\right\}$$

- The number of estimates "explodes" (results are unreliable):
  - $N = 50$ securities $\Longrightarrow$ 1,225 covariance estimates (only 100 σ and μ estimates)
  - $N = 500$ securities $\Longrightarrow$ 124,750 covariance estimates

Systematic $\times$ Firm-Specific Risk



Source: "Statman (1987) - *How many stocks make a diversified portfolio*"

Index Model: Structure

- Decomposing returns into two components:

$$r_{i,t} - r_f = \beta_i \cdot \underbrace{(r_{M,t} - r_f)}_{systematic} + \underbrace{\alpha_i + e_{i,t}}_{firm-specific}$$

$$\Downarrow$$

$$\mathbb{E}[r_{i,t}] = r_f + \alpha_i + \beta_i \cdot (\mathbb{E}[r_{M,t}] - r_f)$$

$$\sigma^2[r_{i,t}] = \beta_i^2 \cdot \sigma^2[r_{M,t}] + \sigma^2[e_{i,t}]$$

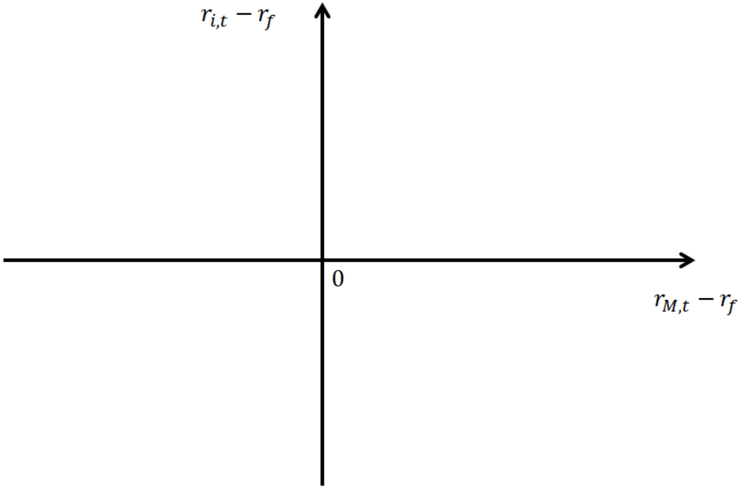
$$Cov[r_A, r_B] = Cov[\beta_A \cdot (r_M - r_f) + e_A, \beta_B \cdot (r_M - r_f) + e_B]$$

$$= \beta_A \cdot \beta_B \cdot \sigma^2[r_{M,t}]$$

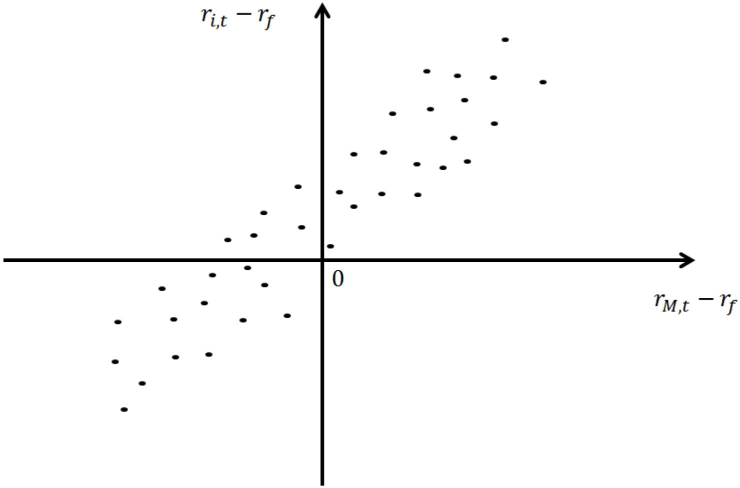
Index Model: Reduction in Number of Estimates

- With an index model with N assets, we need:
  - N estimates of α_i, β_i and $\sigma\left[e_{i,t}\right]$
  - 1 estimate $\mathbb{E}\left[r_{M,t}\right]$ and $\sigma^2\left[r_{M,t}\right]$
- The reduction in the number of estimates is substantial:
  - $N = 50$ securities $\Longrightarrow$ 1,355 estimates
  - $N = 50$ securities $\Longrightarrow$ 152 estimates (with index model)

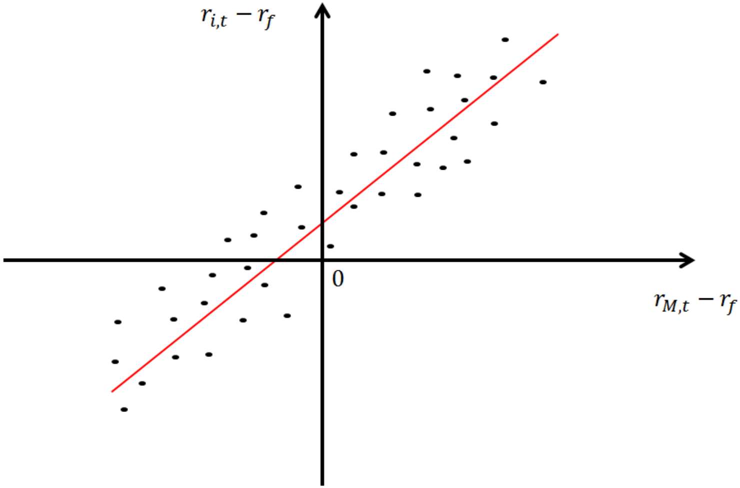
Index Model as a Regression



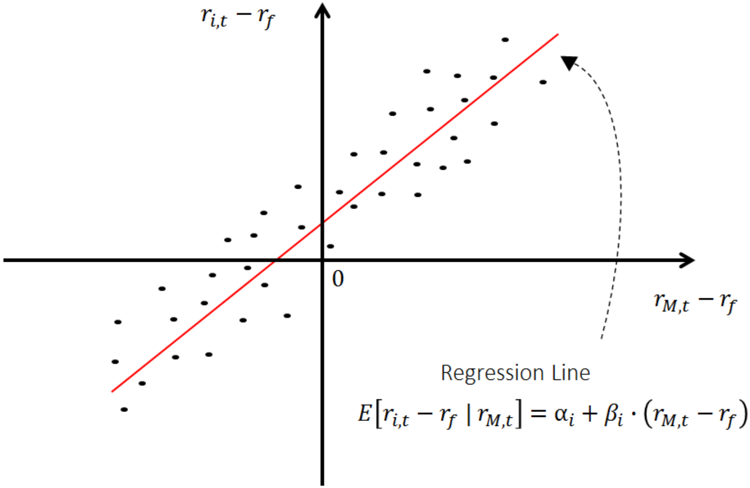
Index Model as a Regression



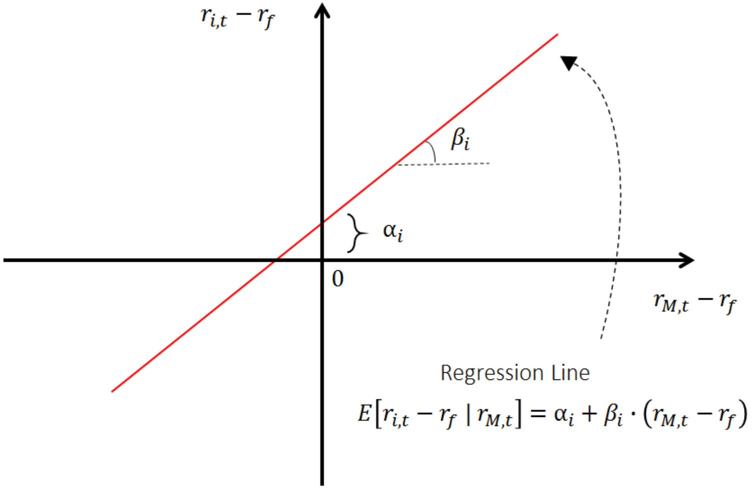
Index Model as a Regression



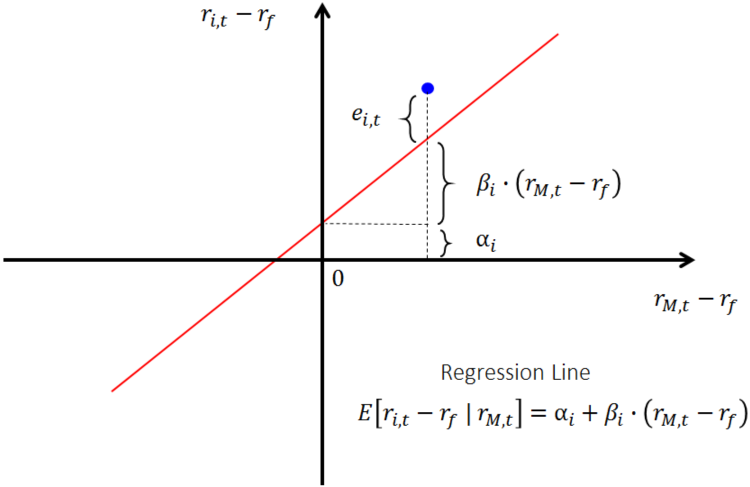
Index Model as a Regression



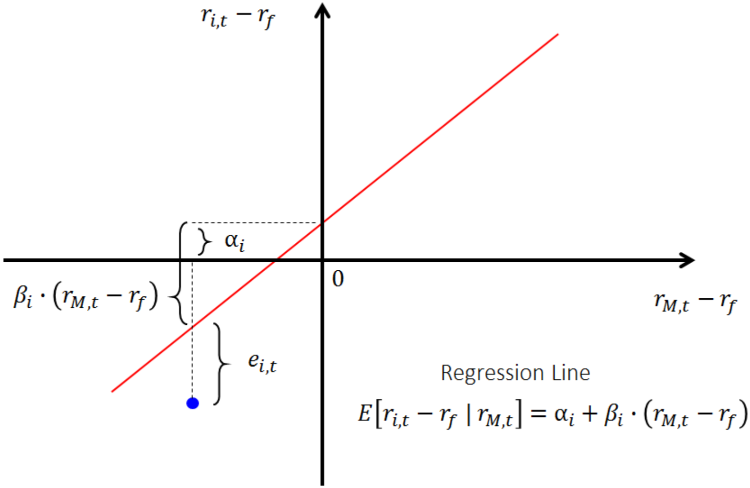
Index Model as a Regression: α_i and β_i



Index Model as a Regression: Decomposing $r_{i,t}$



Index Model as a Regression: Decomposing $r_{i,t}$



Index Model: Systematic × Firm-Specific Risk

- Decomposing risk into two components:

$$\sigma^2\left[r_{i,t}\right] = \underbrace{\beta_i^2 \cdot \sigma^2\left[r_{M,t}\right]}_{systematic\ risk} \quad + \quad \underbrace{\sigma^2\left[e_{i,t}\right]}_{firm-specific\ risk}$$

$$\Downarrow$$

$$R^2 = \frac{\beta_i^2 \cdot \sigma^2\left[r_{M,t}\right]}{\sigma^2\left[r_{i,t}\right]}$$

$$= \frac{\beta_i^2 \cdot \sigma^2\left[r_{M,t}\right]}{\beta_i^2 \cdot \sigma^2\left[r_{M,t}\right] + \sigma^2\left[e_{i,t}\right]}$$

Index Model: Systematic × Firm-Specific Risk

- When we form a (equal-weighted) portfolio $r_p = \frac{1}{N}\sum_{i=1}^{N} r_i$:

$$r_p - r_f = \underbrace{\left(\frac{1}{N}\sum_i \alpha_i\right)}_{\alpha_p} + \underbrace{\left(\frac{1}{N}\sum_i \beta_i\right)}_{\beta_p} \cdot (r_M - r_f) + \left(\frac{1}{N}\sum_i e_i\right)$$

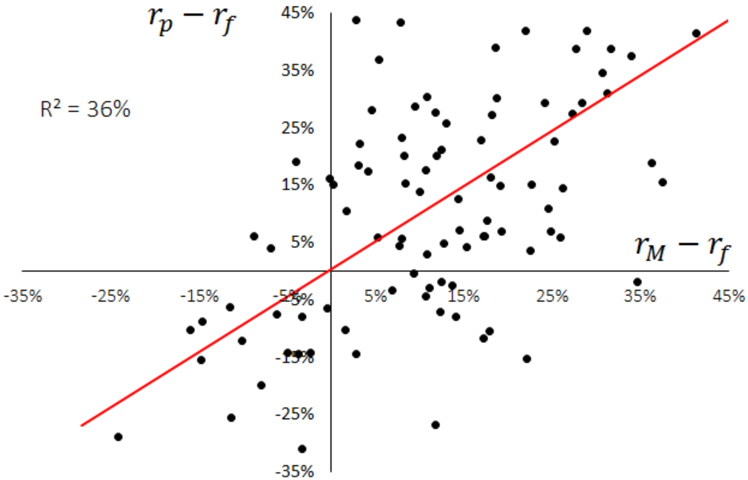
$$= \alpha_p + \beta_p \cdot (r_M - r_f) + \left(\frac{1}{N}\sum_i e_i\right)$$

$$\cong \alpha_p + \beta_p \cdot (r_M - r_f)$$

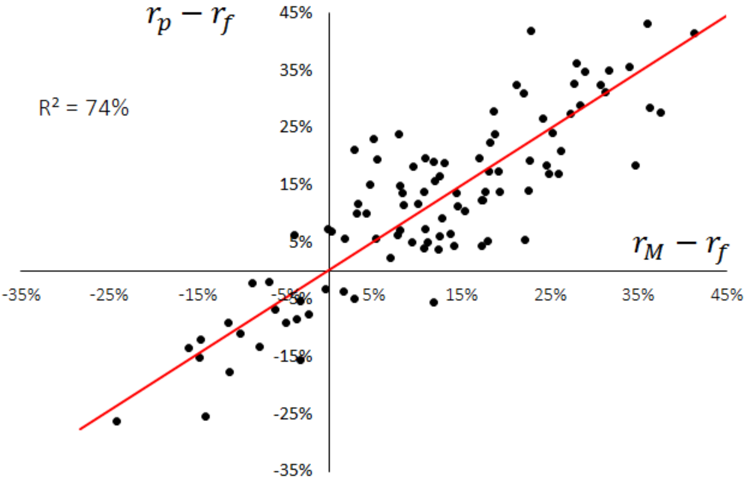
$$\Downarrow$$

$$\sigma^2 [r_p] \cong \beta_p^2 \cdot \sigma^2 [r_M]$$

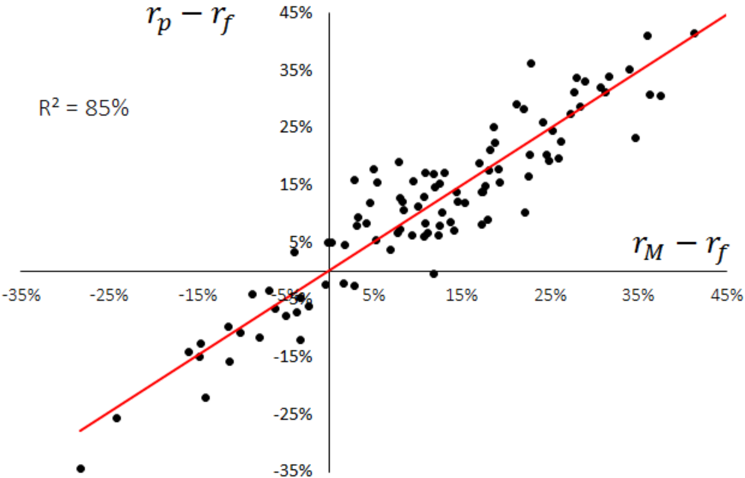
$N = 1$



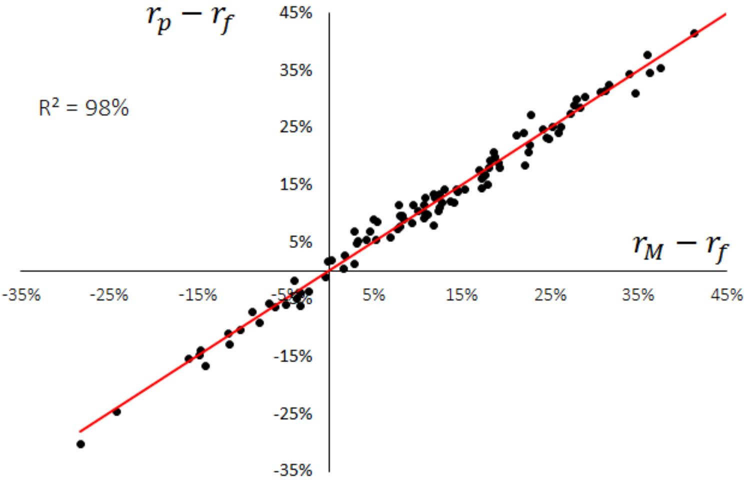
$N = 5$



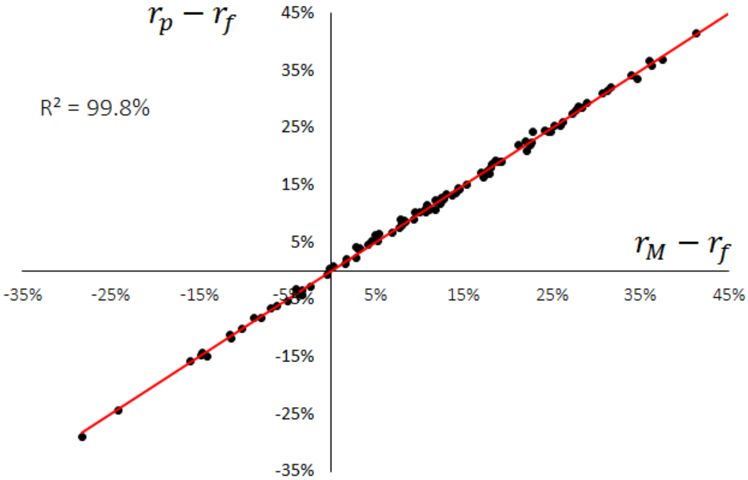
$N = 10$



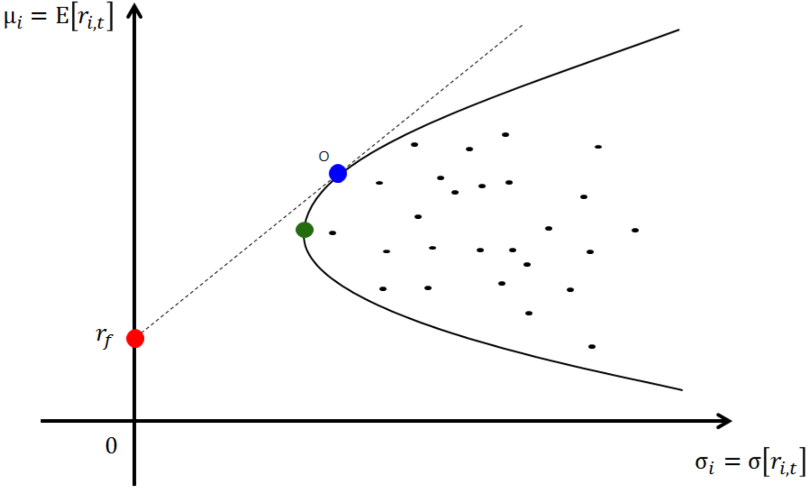
$N = 100$



$N = 1,000$



Index Model: Finding the Tangency Portfolio



Index Model: Finding the Tangency Portfolio

- It turns out that (using the index model):

$$r_o = \underbrace{w_A \cdot r_A}_{Active} + \underbrace{(1 - w_A) \cdot r_M}_{Passive}$$

with

$$w_A^0 = \frac{\alpha_A / \sigma^2[e_A]}{(\mathbb{E}[r_M] - r_f) / \sigma_M^2} \quad and \quad w_A = \frac{w_A^0}{1 + w_A^0 \cdot (1 - \beta_A)}$$

- Moreover, weight of asset i in the active portfolio is:

$$w_i^A = \frac{\alpha_i / \sigma^2[e_i]}{\sum_i \alpha_i / \sigma^2[e_i]}$$

You are considering using an index model when estimating inputs for your portfolio optimization problem. All of the followings are advantages of this approach of estimating inputs, except:

a) It reduces substantially the number of parameters to be estimated

b) It creates a clear decomposition between systematic and firm-specific risk

c) It breaks the optimal risky portfolio into a passive portfolio and an active position, which allows you to understand how you are deviating from the given index

d) It provides you with estimates that rely on a lower number of assumptions about returns

e) It provides a simple way to incorporate the fact that many events in the market affect several assets simultaneously

A Word of Caution Regarding Portfolio Theory

- Portfolio theory provides you with an extremely useful tool for portfolio formation. However:
  - "Garbage in, garbage out" principle
  - Portfolio Theory is silent about how to estimate $\mathbb{E}[r_t]$ and $\sigma[r_t]$. Forward looking estimates are key (security analysis)
  - Maximum Sharpe Ratio portfolio is very sensitive to model inputs (especially $\mathbb{E}[r_t]$)
  - The reality of institutional details needs to be incorporated (no short-sale; no leverage; portfolio similar to benchmark...)
  - Tail risk and other sources of risk are not being taken into account in the standard procedure

References