Module 2: Portfolio Theory

(BUSFIN 4221 - Investments)

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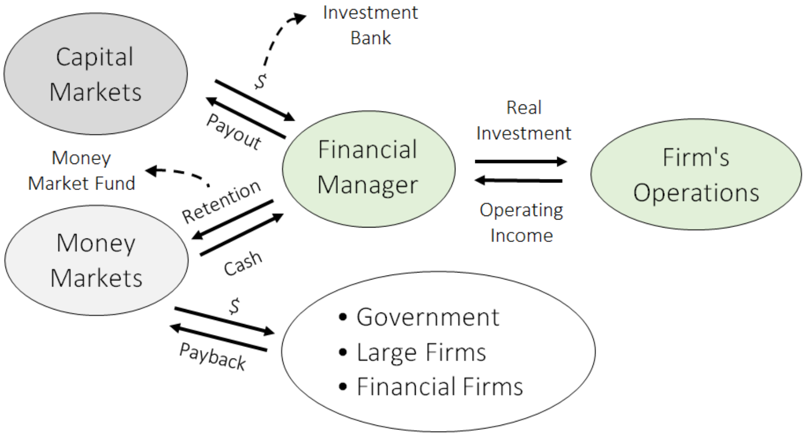
The Ohio State University

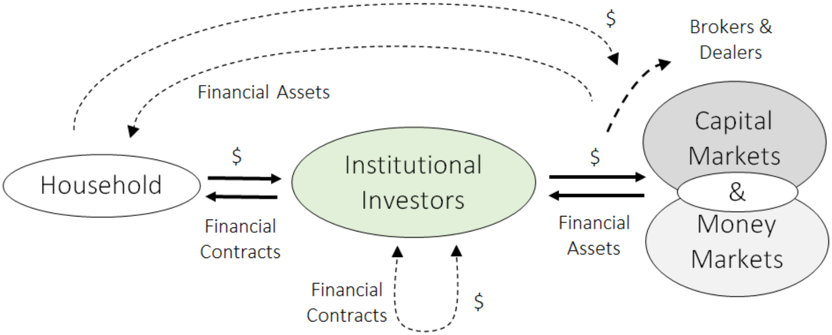
Fall 2016

Outline

Module 1 - The Demand for Capital



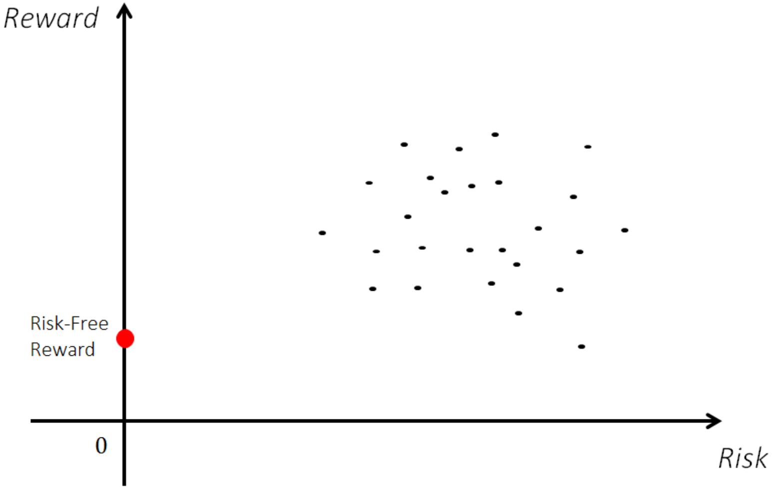
Module 1 - The Supply of Capital



Module 1 - Investment Principle

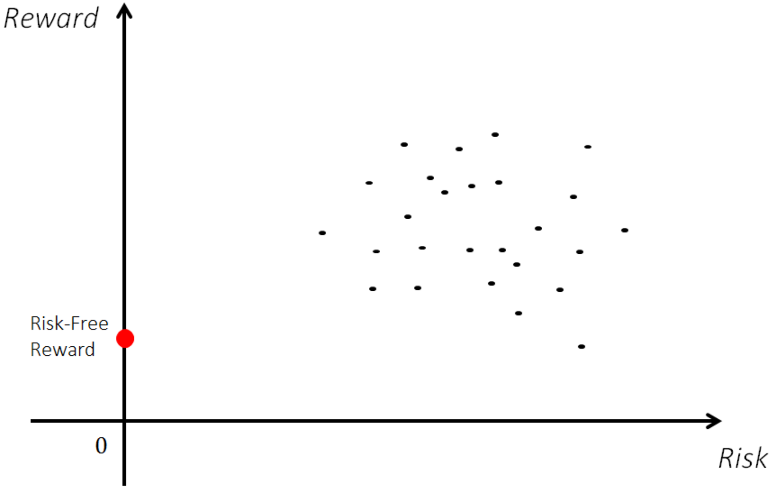
$$PV_t = \sum_{h=1}^{\infty} \frac{\mathbb{E}_t \left[CF_{t+h} \right]}{\left(1 + dr_{t,h}\right)^h}$$

This Module: Creating a Portfolio



Outline

This Section: Defining Risk and Reward



Measuring Performance: Returns

$$r_t = \frac{(P_t + CF_t) - P_{t-1}}{P_{t-1}}$$

$$= \underbrace{\frac{(P_t - P_{t-1})}{P_{t-1}}}_{Capital\ Gain} + \underbrace{\frac{CF_t}{P_{t-1}}}_{Yied}$$

- Arithmetic average return (or simple "average return"):

$$\bar{r} = \frac{r_1 + r_2 + \ldots + r_T}{T}$$

- Geometric average return:

$$\bar{r}_G = \{(1 + r_1) \times (1 + r_2) \times \ldots \times (1 + r_T)\}^{1/T}$$

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Measuring Performance: Annual Returns

- A return of 1% in the previous month is not comparable with a return of 12% in the previous year. We need to fix the time period of different returns to make them comparable

- The effective annual rate, ear_t, does that for you:

$$1 + ear_t = (1 + r_t)^n$$

- n is the number of periods in a year. For instance, $n = 1$ for annual r_t and $n = 12$ for monthly r_t

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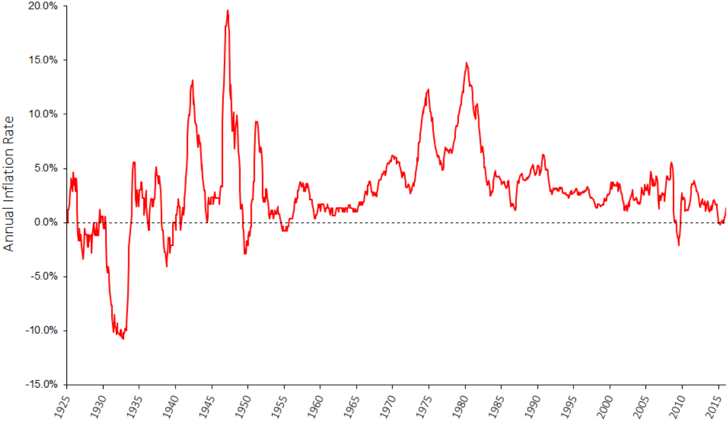
Measuring Performance: Inflation Effect



$$1 + r_t^{real} = \frac{1 + r_t}{1 + i_t} \qquad \Leftrightarrow \qquad r_t^{real} \cong r_t - i_t$$

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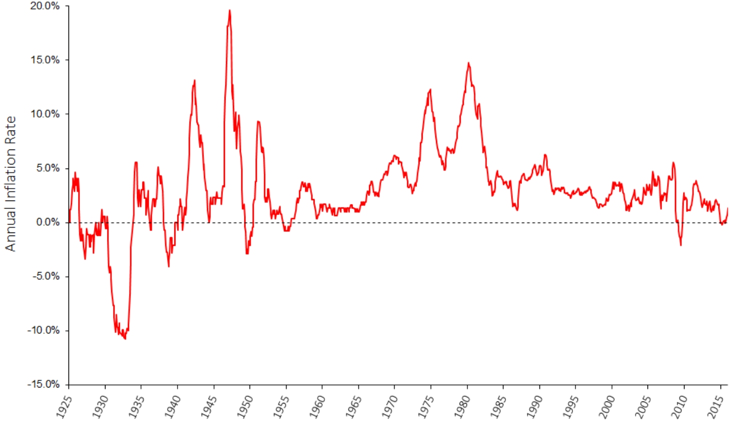

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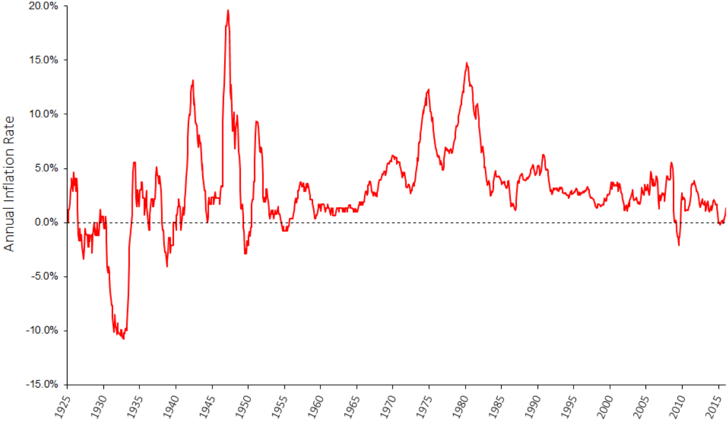

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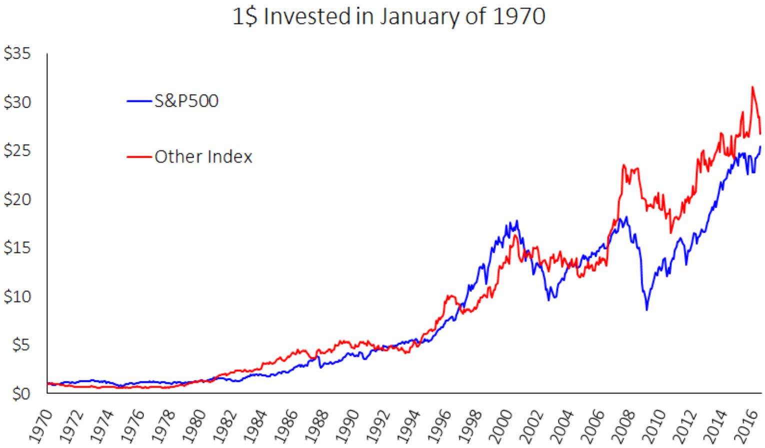
Returns as a Random Variable: Indices



Returns as a Random Variable: $\mathbb{E}[r_t]$ and $\sigma[r_t]$

Economic Scenario Next Year (s)	$p(s)$	$r(s)$	$r(s) - \mathbb{E}[r_t]$
Very High Growth	0.15	30%	20%
Growth > $\mathbb{E}[Growth]$	0.25	20%	10%
Growth = $\mathbb{E}[Growth]$	0.35	10%	0%
Growth < $\mathbb{E}[Growth]$	0.2	-5%	-15%
Recession	0.05	-40%	-50%

$$\mathbb{E}[r_t] = \sum_s p(s) \times r(s) = 10\%$$

$$\sigma[r_t] = \sqrt{\sum_s p(s) \times \{r(s) - \mathbb{E}[r_t]\}^2} = 16\%$$

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Recession	0.05	-60%	-70%

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$$\sigma[r_t] = \sqrt{\sum_s p(s) \times \{r(s) - \mathbb{E}[r_t]\}^2} = 26\%$$

Modeling Returns: $r_t \sim \mathcal{N}(\mu, \sigma)$

- $\mu = \mathbb{E}[r_t]$ and $\sigma = \sigma[r_t]$
- Index simulation: I simulate r_t and use it to get future prices
- If r_t are truly normal, then only μ and σ matter (they describe entire distribution). In this case, the only measure of risk is σ
- If $r_{i,t}$ are normal, then so are portfolio returns: $r_{p,t} = w_1 \times r_{1,t} + w_2 \times r_{2,t} + ... + w_N \times r_{N,t}$
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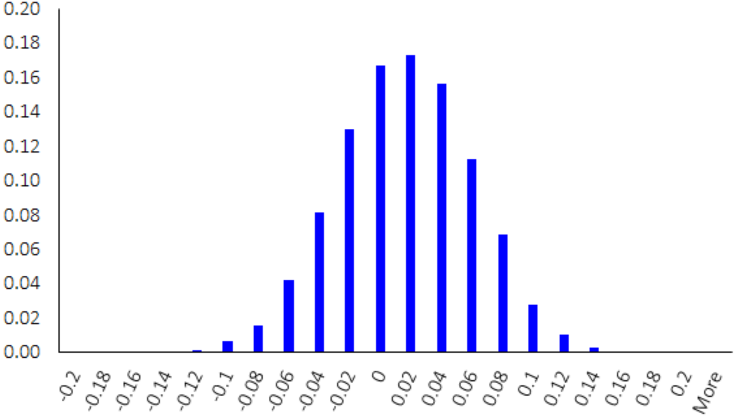
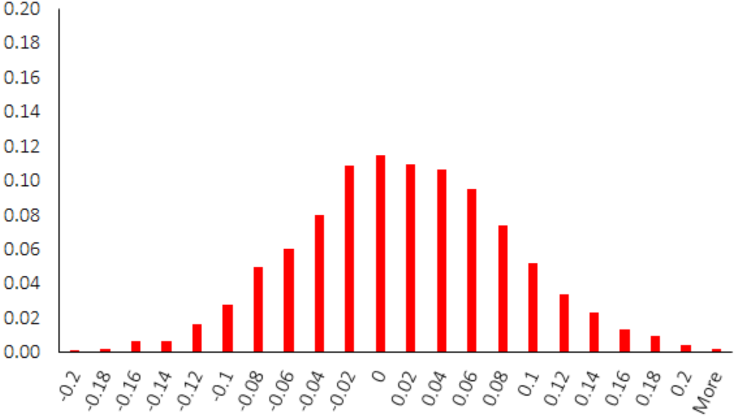
Modeling Returns: $r_t \sim \mathcal{N}(\mu = 0.6\%, \sigma = 4.4\%)$



Modeling Returns: $r_t \sim \mathcal{N}(\mu = 0.6\%, \sigma = 7.0\%)$



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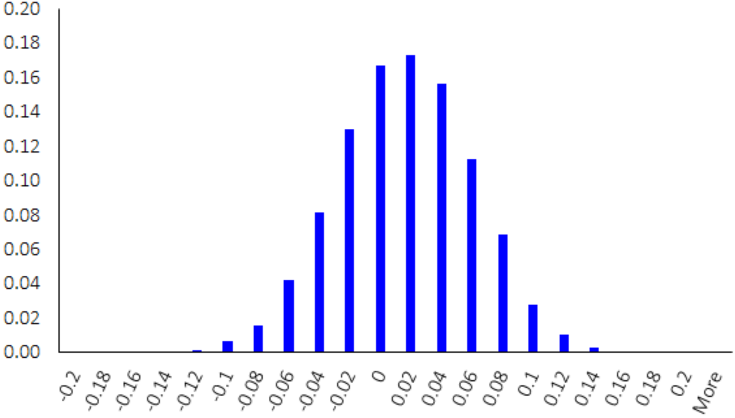
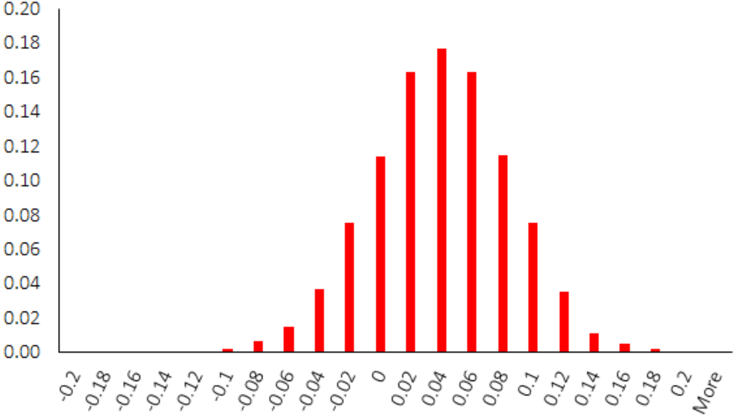
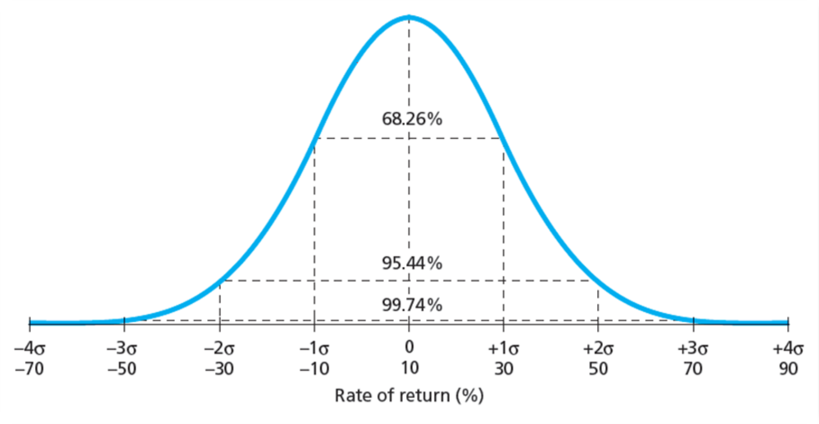


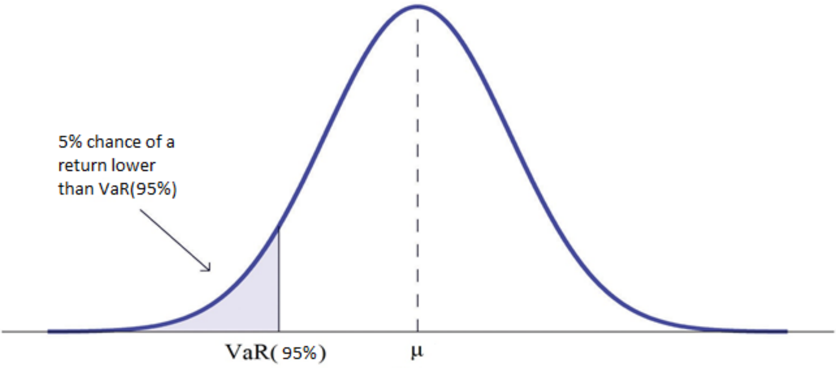
Modeling Returns: $r_t \sim \mathcal{N}(\mu = 3.0\%, \sigma = 4.4\%)$



Modeling Returns: $r_t \sim \mathcal{N}(\mu = 10\%, \sigma = 10\%)$



Modeling Returns: Tail Risk for Normal Returns?



$$VaR\,(95\%) = \mathbb{E}\,[r_t] - 1.64 \times \sigma\,[r_t]$$

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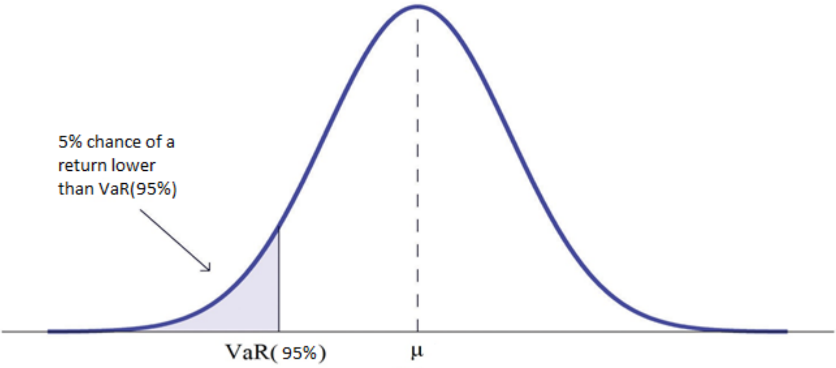

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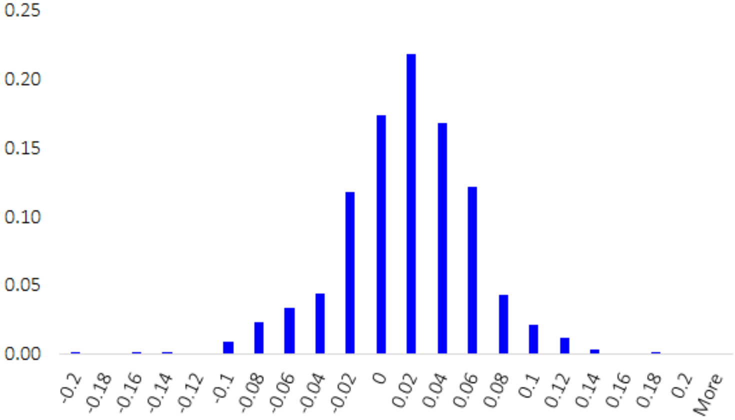
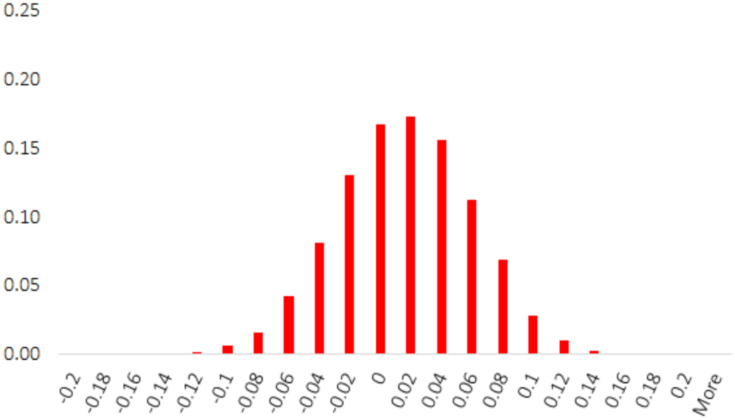
Modeling Returns: S&P 500



Modeling Returns: $r_t \sim \mathcal{N}\left(\mu_{S\&P}, \sigma_{S\&P}\right)$



Data: Estimating Model from Time Series of r_t

- If $r_t \sim \mathcal{N}(\mu, \sigma)$, how can we estimate μ and σ from data?

- Recall: $\mu = \mathbb{E}[r_t] = \sum_s p(s) \times r(s)$

$$\sigma = \sigma[r_t] = \sqrt{\sum_s p(s) \times \{r(s) - \mathbb{E}[r_t]\}^2}$$

- Use each data observation as a "scenario" with equal probability:

$$\hat{\mu} = \frac{1}{T} \times \sum_{t=1}^{T} r_t = \bar{r}$$

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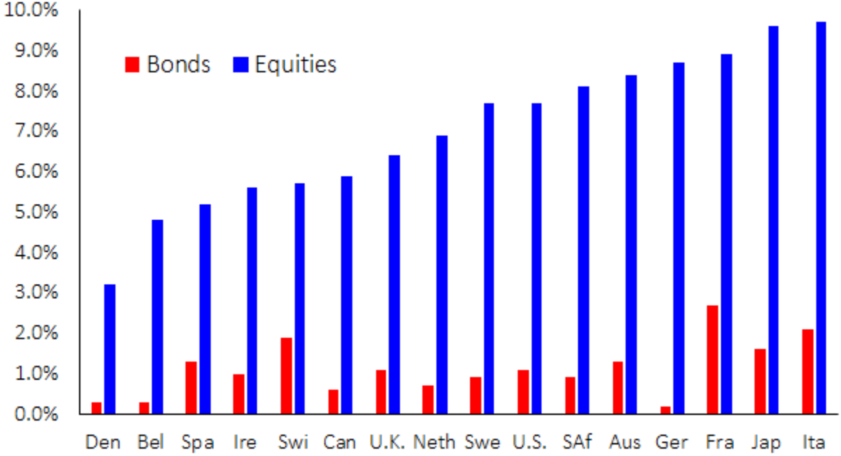
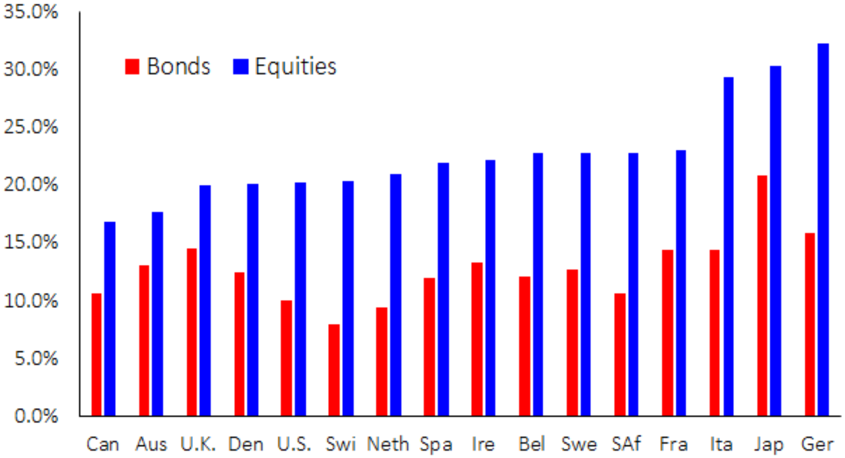
Data: $(\bar{r} - \bar{r}_{TBill})$ around the World from 1900-2000



Source: Dimson et al (2002) - *Triumph of the optimists: 101 years of global investment returns*

Data: $\hat{\sigma}\left[r_t\right]$ around the World from 1900-2000



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If asset A daily r_t follow a normal distribution, then:

a) Asset A monthly r_t also do

b) An increase in $\sigma[r_t]$ implies an increase in the tail risk the asset

c) Inflation does not matter for the real return on asset A

d) The geometric average return and the average return are the same

e) All we need to know about asset A in order to fully understand its daily returns distribution is $\sigma[r_t]$

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Outline

Overview

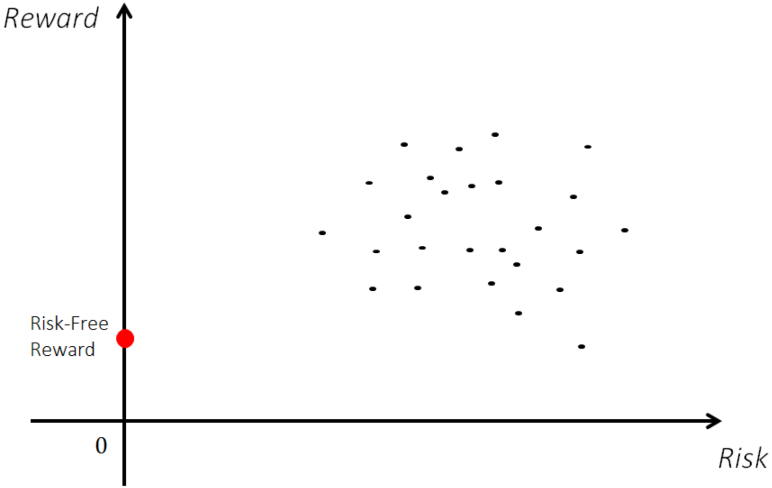
The Statistics of Security Returns

The Efficient Frontier

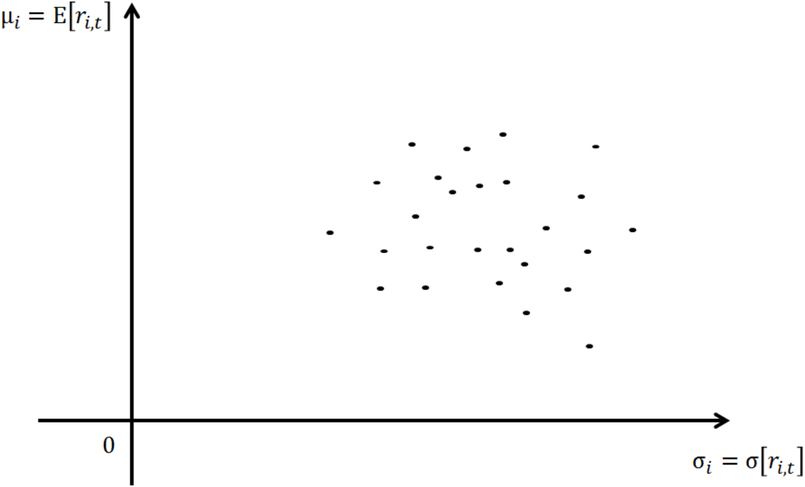
Capital Allocation Line

Index Models

This Section: Find the “Best” Risky Portfolios



$\sigma[r_t] \times \mathbb{E}[r_t]$: Principle



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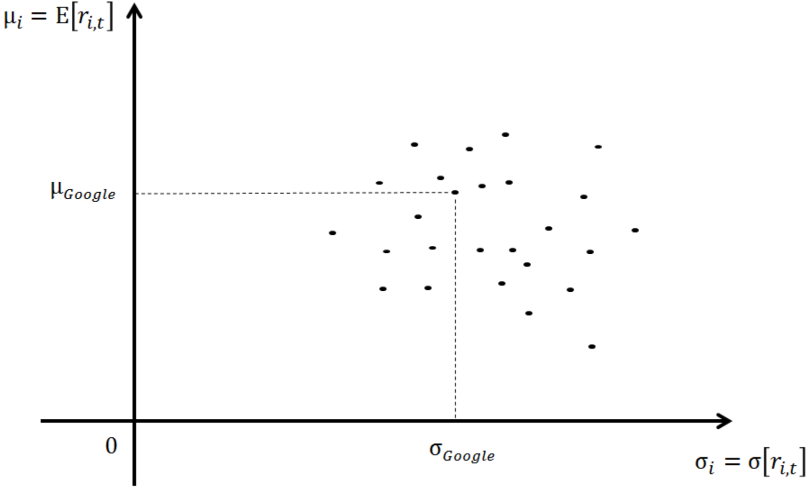

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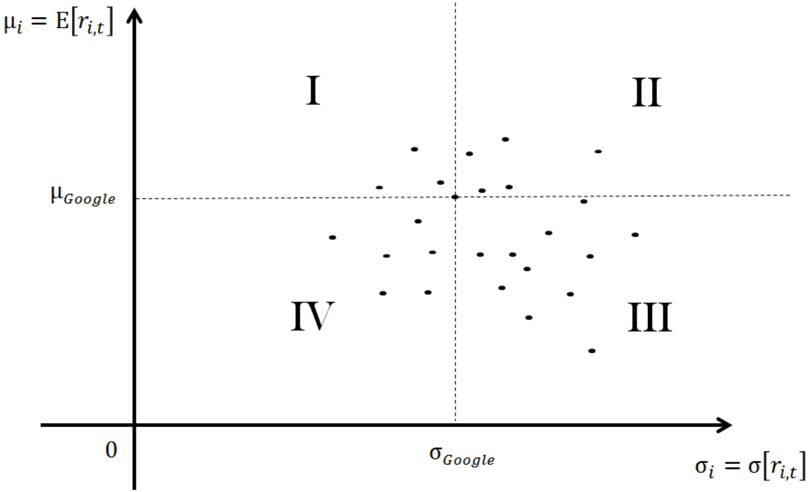

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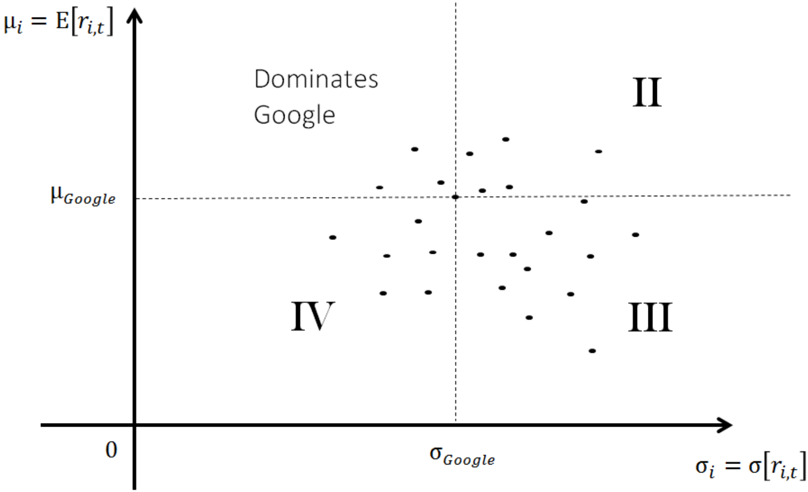

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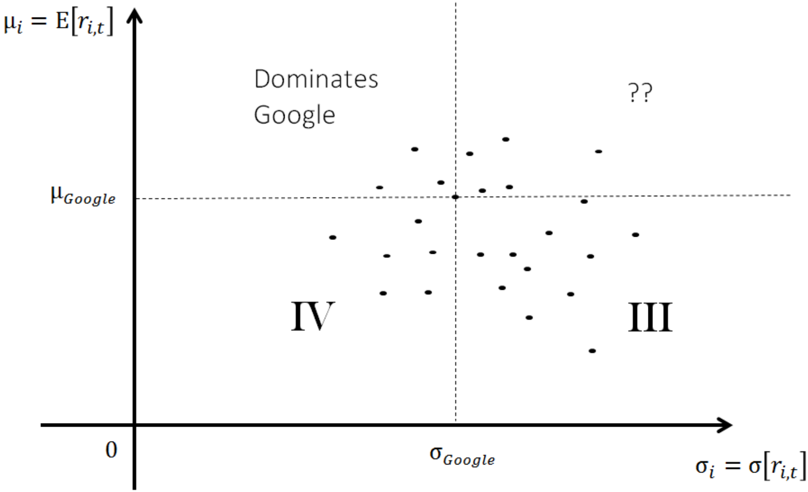

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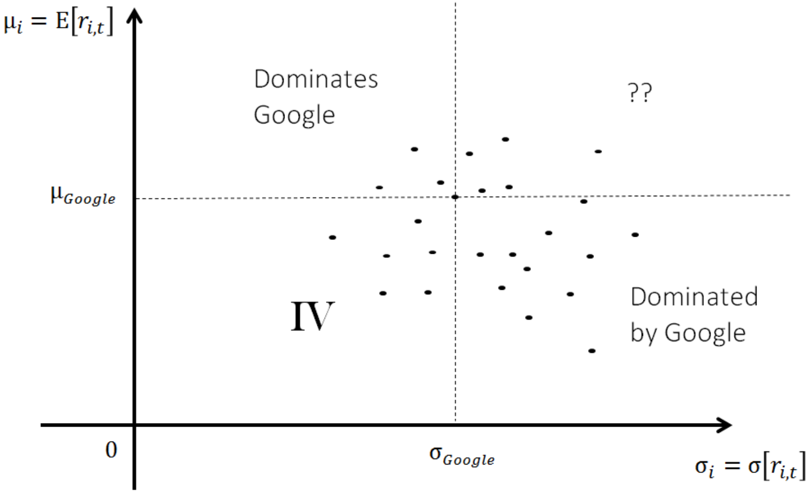

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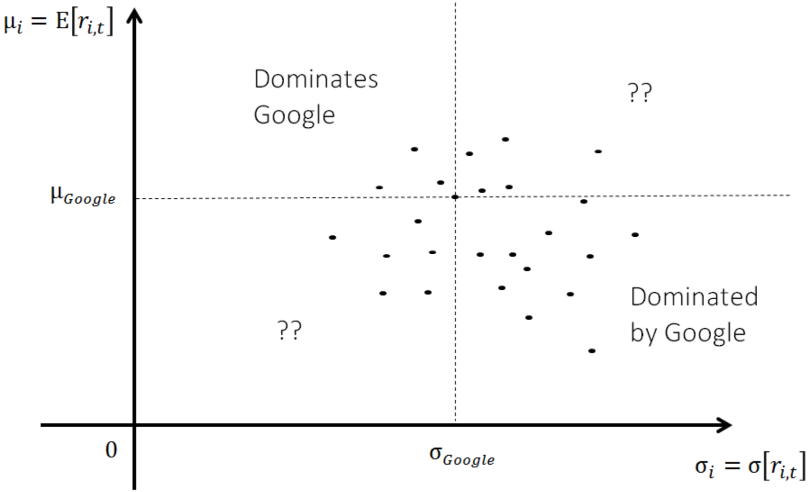

Diversification: Basic Principle

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Diversification: Simple Example

Economic Scenario Next Year (s)	$p(s)$	r_{Equity}	r_{Gold}	50% – 50% $r_{Portfolio}$
Very High Growth	0.15	30%	-12%	9%
Growth $> \mathbb{E}[Growth]$	0.25	20%	5%	13%
Growth $= \mathbb{E}[Growth]$	0.35	10%	15%	13%
Growth $< \mathbb{E}[Growth]$	0.2	-5%	20%	8%
Recession	0.05	-40%	25%	-8%
$\mathbb{E}[r_t] =$				
$\sigma[r_t] =$				

Diversification: Simple Example

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Recession	0.05	-40%	25%	-8%
$\mathbb{E}[r_t] =$		10%	10%	
$\sigma[r_t] =$		16%	11%	

Diversification: Simple Example

Economic Scenario Next Year (s)	$p(s)$	r_{Equity}	r_{Gold}	$50\% - 50\%$ $r_{Portfolio}$
Very High Growth	0.15	30%	-12%	9%
Growth $> \mathbb{E}[Growth]$	0.25	20%	5%	13%
Growth $= \mathbb{E}[Growth]$	0.35	10%	15%	13%
Growth $< \mathbb{E}[Growth]$	0.2	-5%	20%	8%
Recession	0.05	-40%	25%	-8%
$\mathbb{E}[r_t] =$		10%	10%	10%
$\sigma[r_t] =$		16%	11%	5%

Diversification: Simple Example

Economic Scenario Next Year (s)	$p(s)$	r_{Equity}	r_{Gold}	$40\% - 60\%$ $r_{Portfolio}$
Very High Growth	0.15	30%	-12%	5%
Growth $> \mathbb{E}[Growth]$	0.25	20%	5%	11%
Growth $= \mathbb{E}[Growth]$	0.35	10%	15%	13%
Growth $< \mathbb{E}[Growth]$	0.2	-5%	20%	10%
Recession	0.05	-40%	25%	-1%
$\mathbb{E}[r_t] =$		10%	10%	10%
$\sigma[r_t] =$		16%	11%	4%

Diversification: Simple Example

Economic Scenario Next Year (s)	$p(s)$	r_{Equity}	r_{Gold}	$50\% - 50\%$ $r_{Portfolio}$
Very High Growth	0.15	30%	-12%	9%
Growth $> \mathbb{E}[Growth]$	0.25	20%	5%	13%
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Diversification: Simple Example

Economic Scenario Next Year (s)	$p(s)$	r_{Equity}	r_{Gold}	50% – 50% $r_{Portfolio}$
Very High Growth	0.15	30%	-7%	12%
Growth $> \mathbb{E}[Growth]$	0.25	20%	5%	13%
Growth $= \mathbb{E}[Growth]$	0.35	10%	15%	13%
Growth $< \mathbb{E}[Growth]$	0.2	-5%	26%	11%
Recession	0.05	-40%	-10%	-25%
$\mathbb{E}[r_t] =$		10%	10%	10%
$\sigma[r_t] =$		16%	11%	8%

Diversification: Covariance & Correlation

Economic Scenario Next Year (s)	$p(s)$	r_{Equity}	r_{Gold}	$(r_E - \mathbb{E}[r_E]) \times (r_G - \mathbb{E}[r_G])$
Very High Growth	0.15	30%	-7%	-0.034
Growth $> \mathbb{E}[Growth]$	0.25	20%	5%	-0.005
Growth $= \mathbb{E}[Growth]$	0.35	10%	15%	0.000
Growth $< \mathbb{E}[Growth]$	0.2	-5%	26%	-0.024
Recession	0.05	-40%	-10%	0.100
$\mathbb{E}[r_t] =$		10%	10%	
$\sigma[r_t] =$		16%	11%	

$$Cov[r_E, r_G] = \sum_s p(s) \times \{r_E(s) - \mathbb{E}[r_E]\} \times \{r_G(s) - \mathbb{E}[r_G]\}$$

$$\rho[r_E, r_G] = \frac{Cov[r_E, r_G]}{\sigma[r_E] \times \sigma[r_G]} = -0.34$$

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Diversification: Two Risky Assets

- Consider the portfolio $r_p = w_A \cdot r_A + w_B \cdot r_B$

- Portfolio expected return, $\mathbb{E}[r_p]$, is given by:

$$\mathbb{E}[r_p] = w_A \cdot \mathbb{E}[r_A] + w_B \cdot \mathbb{E}[r_B]$$

- Portfolio variance, $\sigma^2[r_p] = \sigma_p^2$, is given by:

$$\sigma_p^2 = w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2 \cdot w_A \cdot w_B \cdot \mathrm{Cov}[r_A, r_B]$$

$$= w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2 \cdot w_A \cdot w_B \cdot \sigma_A \cdot \sigma_B \cdot \rho[r_A, r_B]$$

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Diversification: Two Risky Assets

- When do we have diversification: $\sigma_p < w_A \cdot \sigma_A + w_B \cdot \sigma_B$?

- Suppose $\rho[r_A, r_B] = 1$, then:

$$\sigma_p^2 = w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2 \cdot w_A \cdot w_B \cdot \sigma_A \cdot \sigma_B \cdot 1$$

$$= (w_A \cdot \sigma_A + w_B \cdot \sigma_B)^2$$

$$\Downarrow$$

$$\sigma_p = w_A \cdot \sigma_A + w_B \cdot \sigma_B$$

- Any $\rho[r_A, r_B] < 1$ produces $\sigma_p < w_A \cdot \sigma_A + w_B \cdot \sigma_B$

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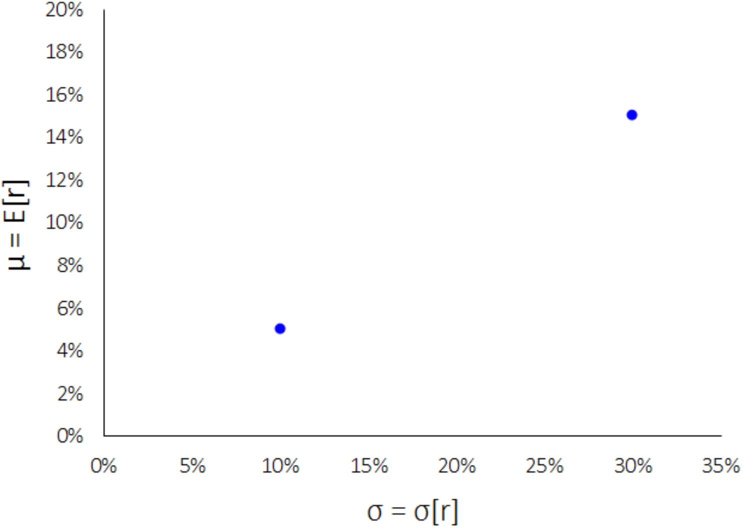
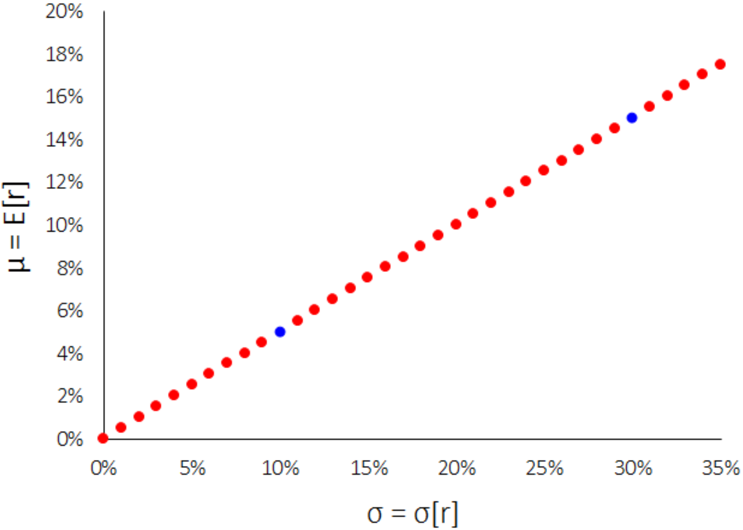
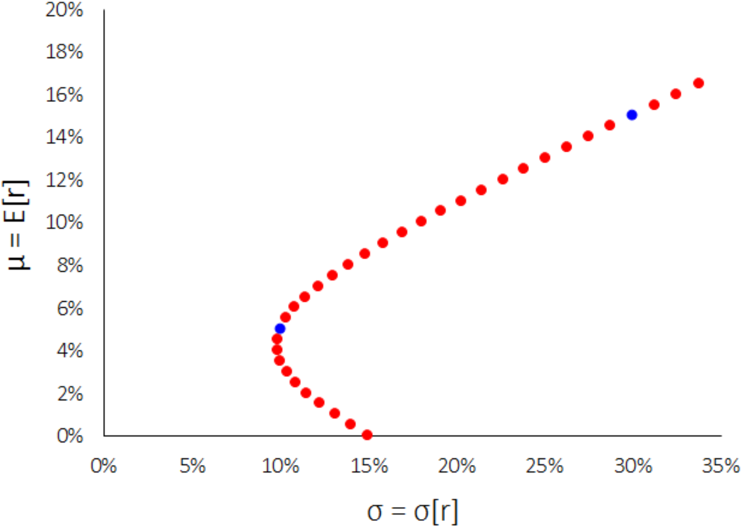
Diversification: Two Risky Assets



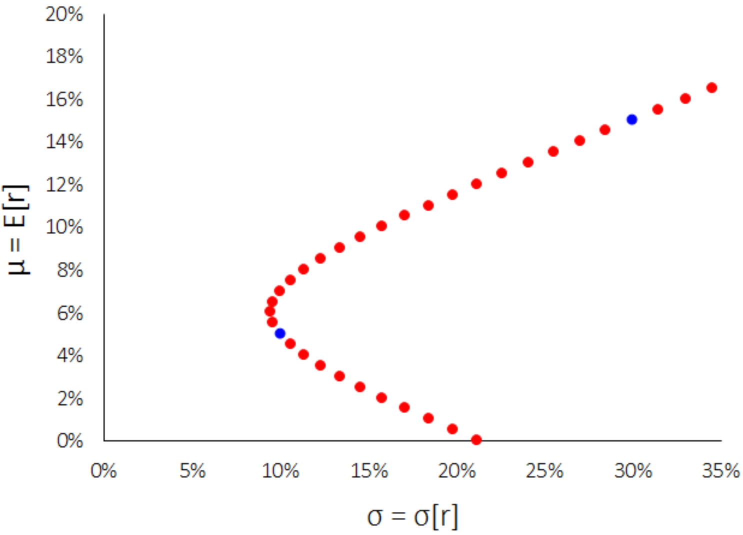
Diversification: Two Risky Assets ($\rho = 1.0$)



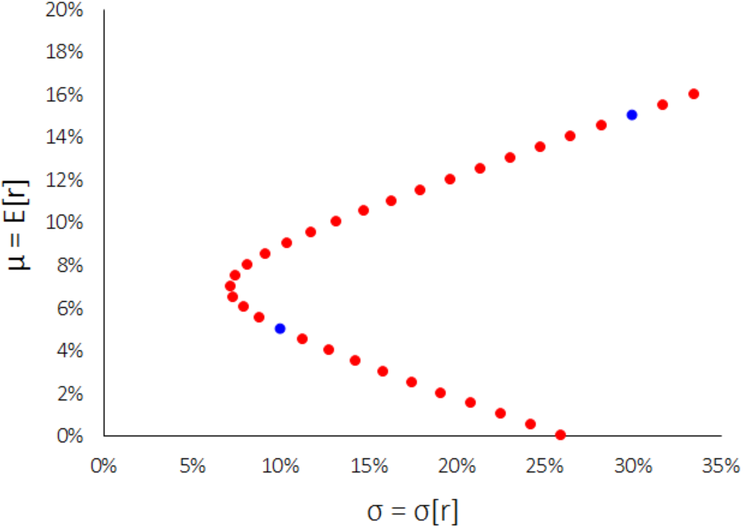
Diversification: Two Risky Assets ($\rho = 0.5$)



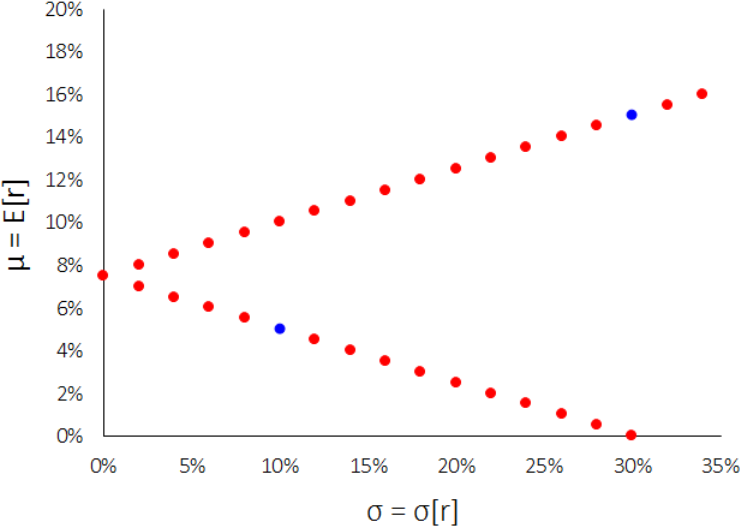
Diversification: Two Risky Assets ($\rho = 0.0$)



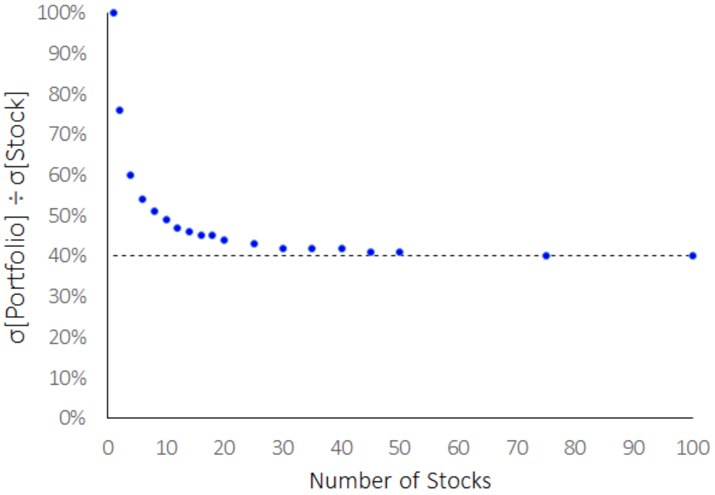
Diversification: Two Risky Assets ($\rho = -0.5$)



Diversification: Two Risky Assets ($\rho = -1.0$)



Diversification: Systematic × Firm-Specific Risk



Source: "Statman (1987) - *How many stocks make a diversified portfolio*"

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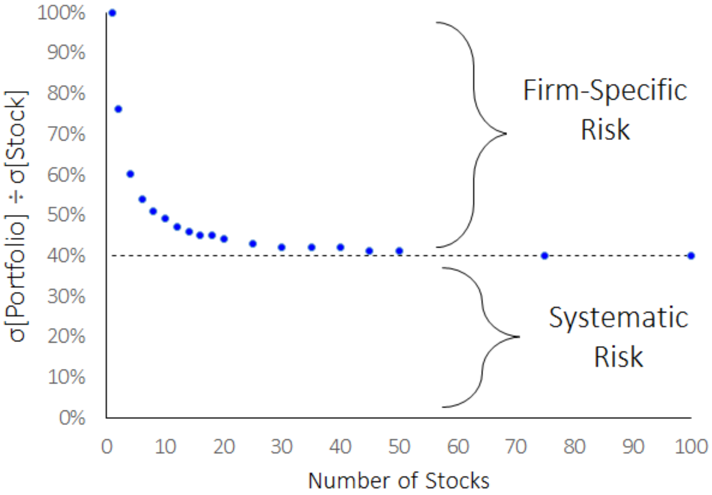

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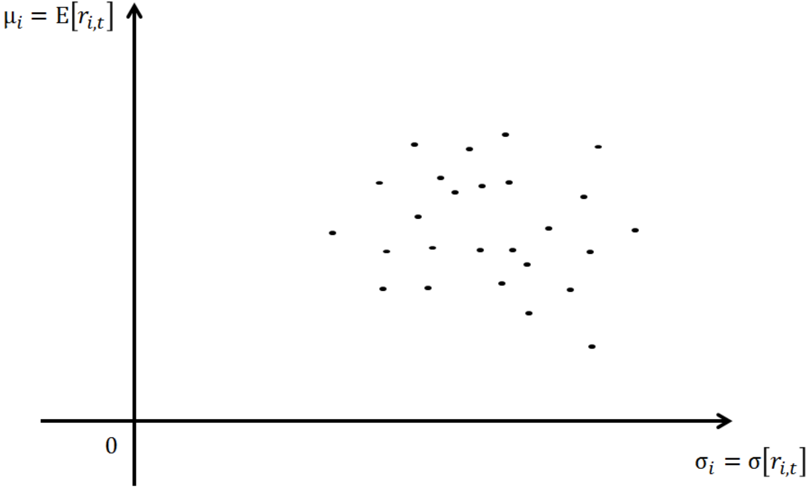
$\sigma[r_t] \times \mathbb{E}[r_t]$: Efficient Frontier



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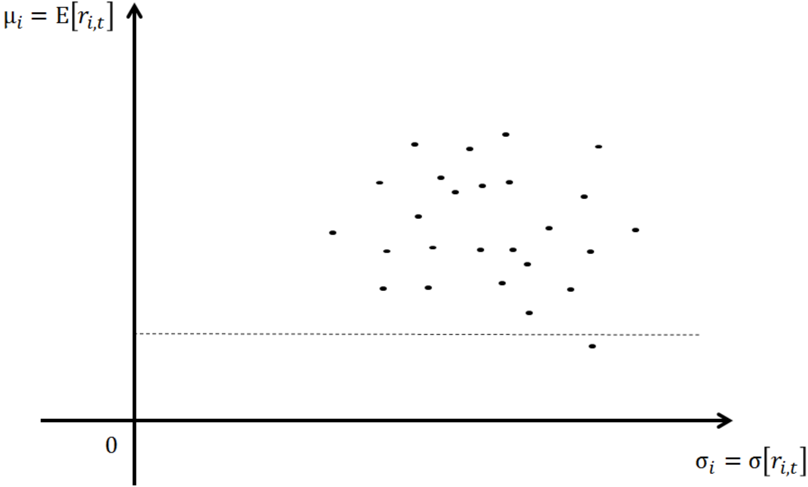

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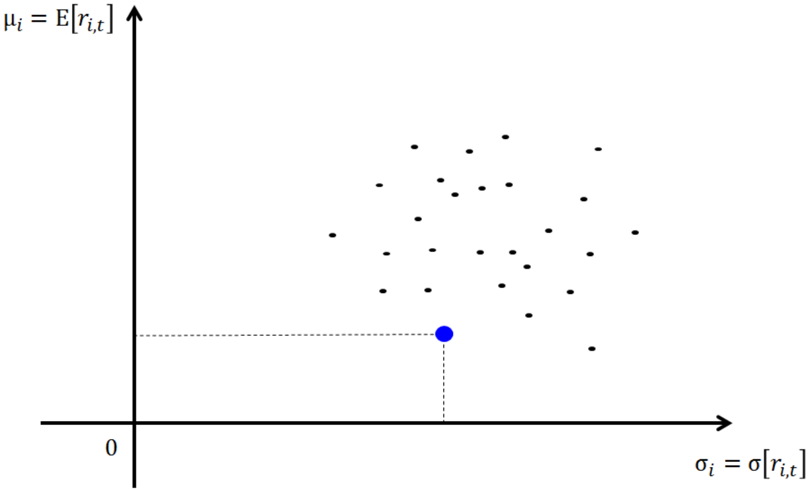

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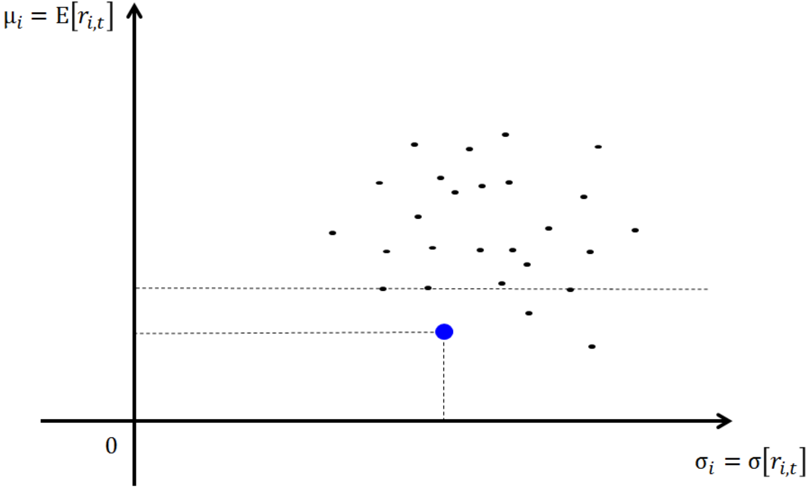

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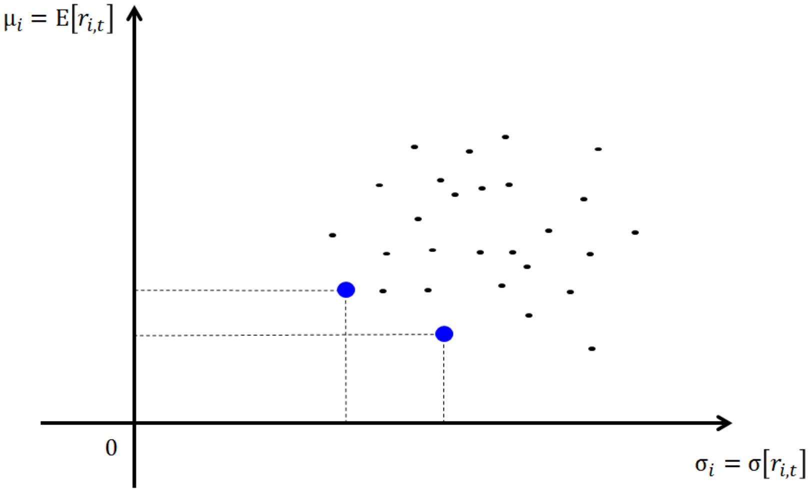

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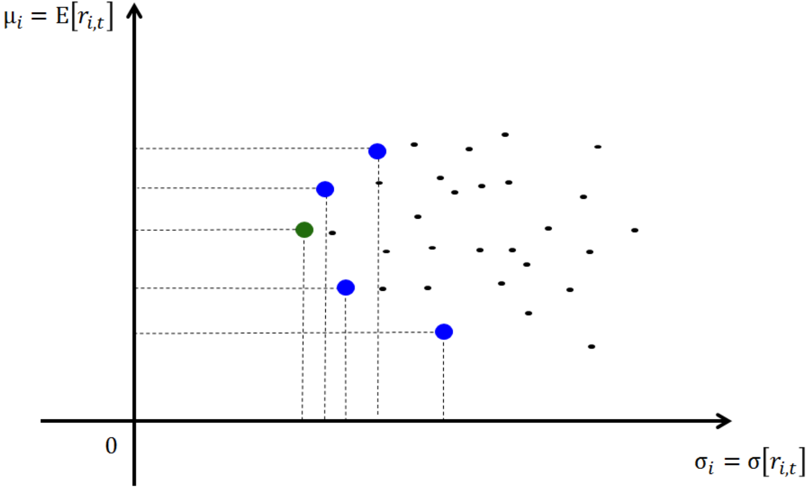

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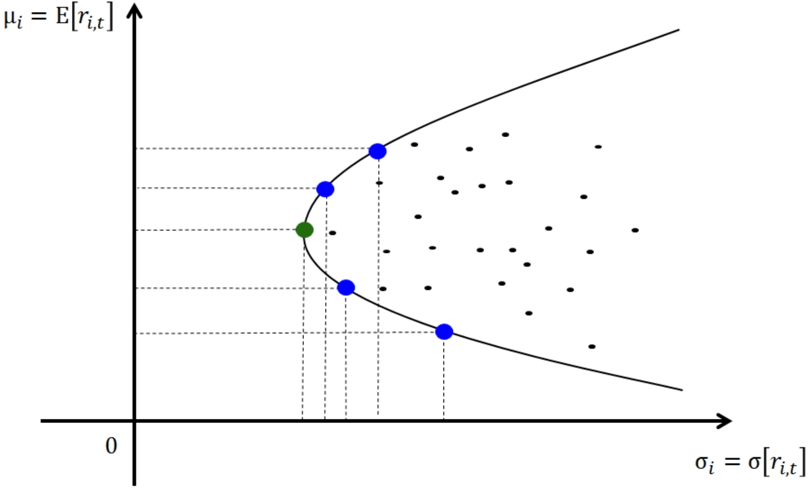

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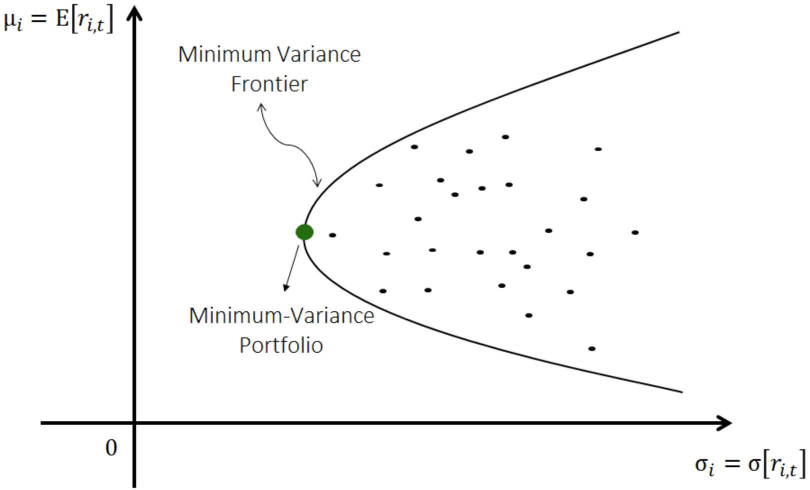

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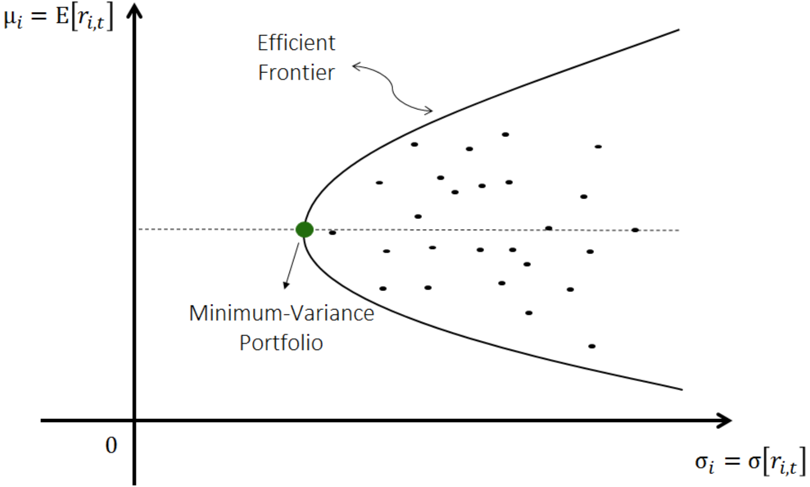

Which of the following is false regarding the Efficient Frontier?

a) It relies on estimates of risk, $\sigma[r_t]$, reward, $\mathbb{E}[r_t]$, and covariances for multiple assets

b) Any portfolio formed by first selecting a target $\mathbb{E}[r_t]$ and then choosing the portfolio with minimum risk among the ones with such target $\mathbb{E}[r_t]$ is an efficient portfolio

c) It restricts the set of potential portfolios an investor should choose from if he measures risk by $\sigma[r_t]$ and reward by $\mathbb{E}[r_t]$

d) It depends heavily on the power of diversification

e) With only two assets, the more negatively correlated they are the better is the efficient frontier investors can form using them

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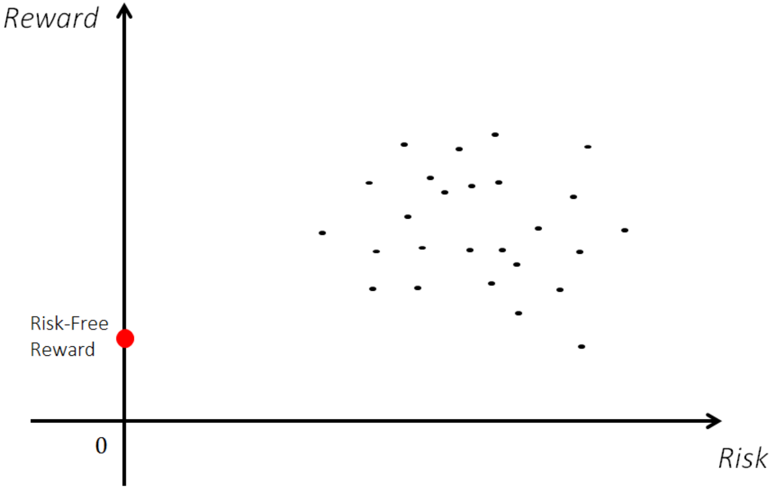
Outline

This Section: Adding Risk Free Asset



Combining Risky Asset with Risk-Free Asset

- Recall that for $r_p = w_A \cdot r_A + w_B \cdot r_B$ we have:

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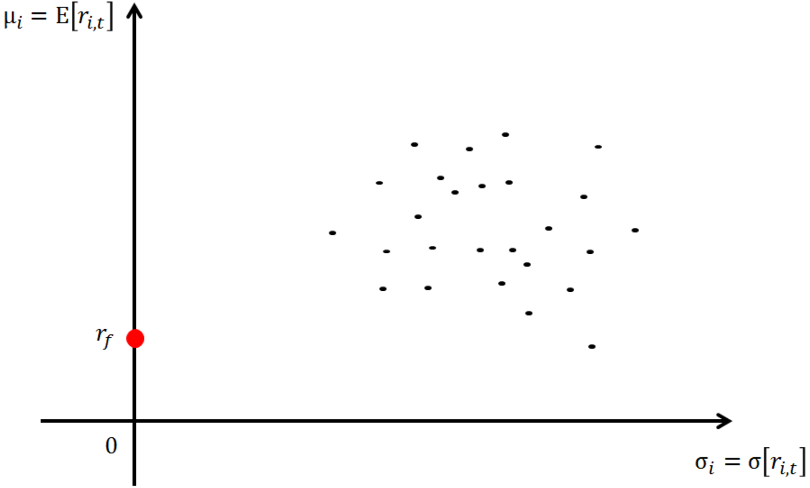

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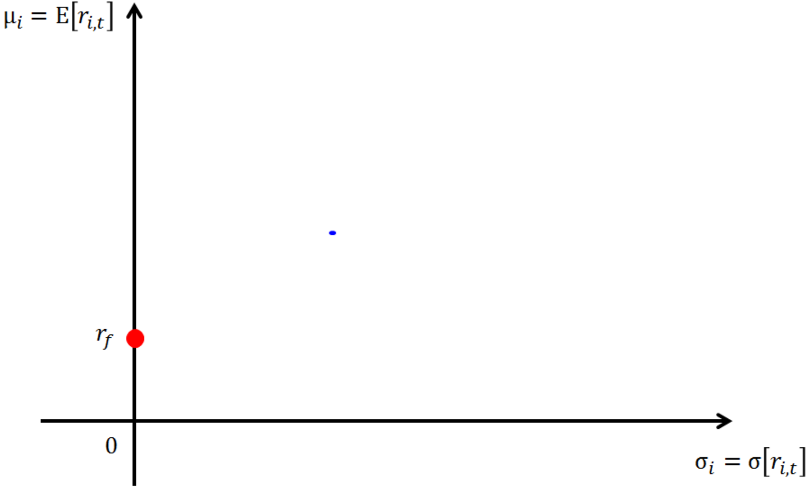

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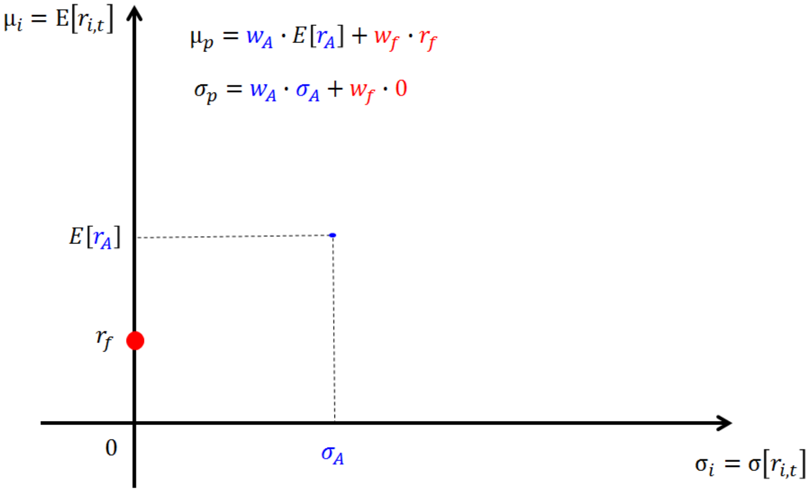

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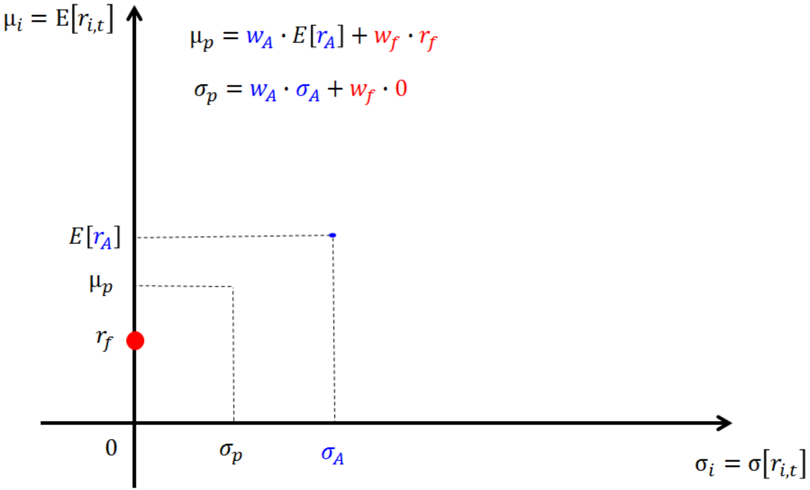

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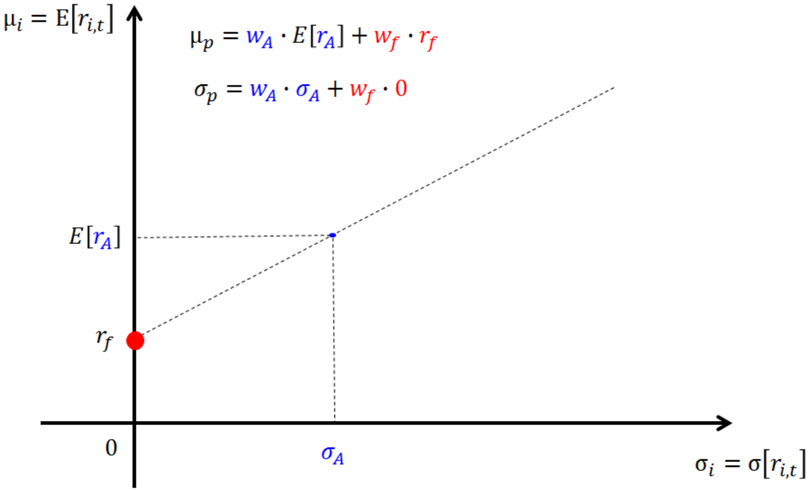
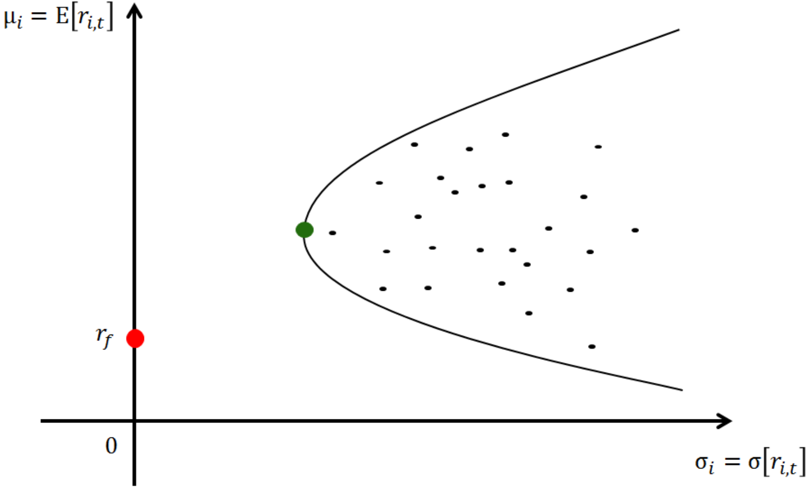


$\sigma[r_t] \times \mathbb{E}[r_t]$: Capital Allocation Line



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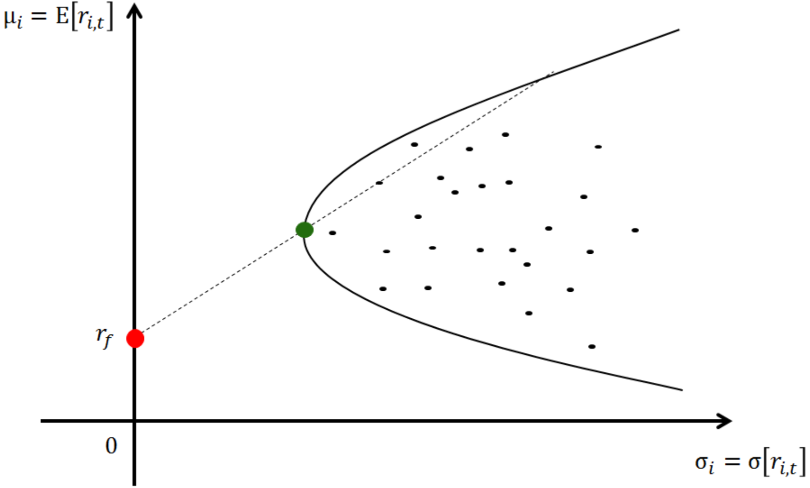

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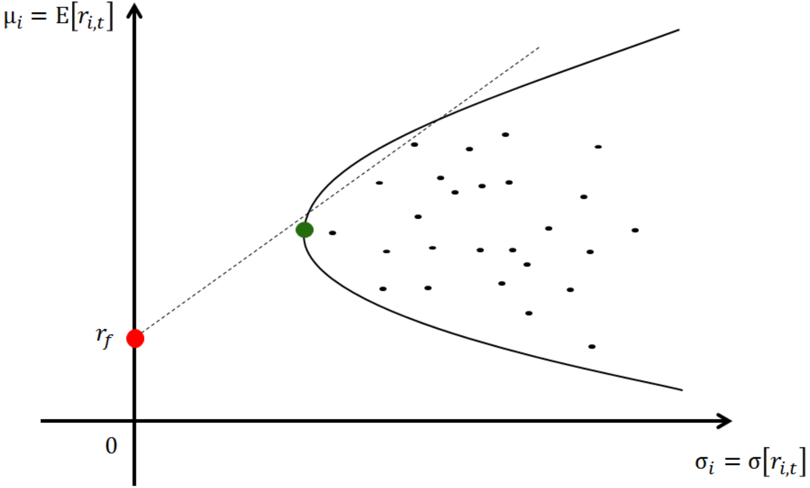

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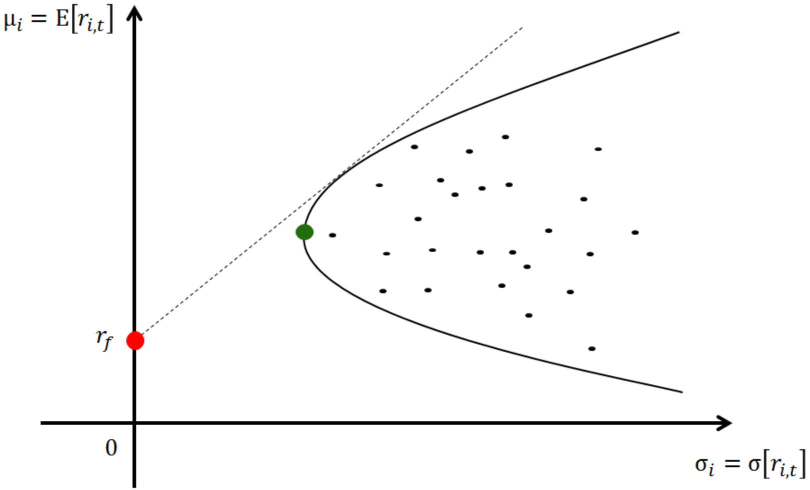

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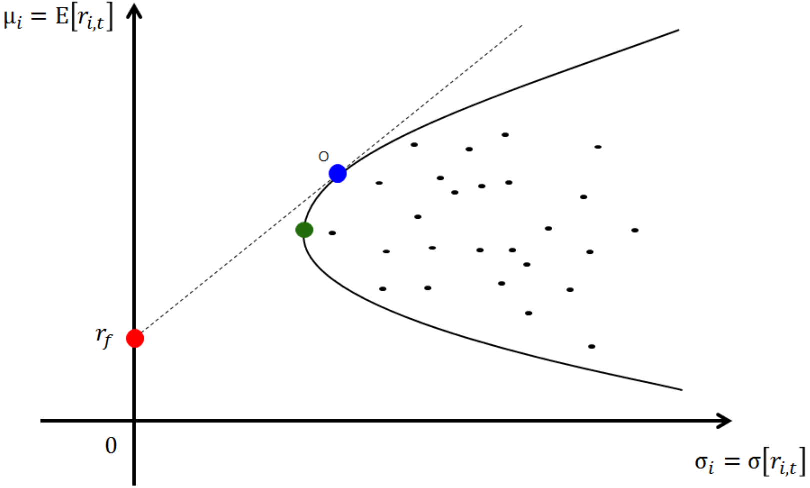

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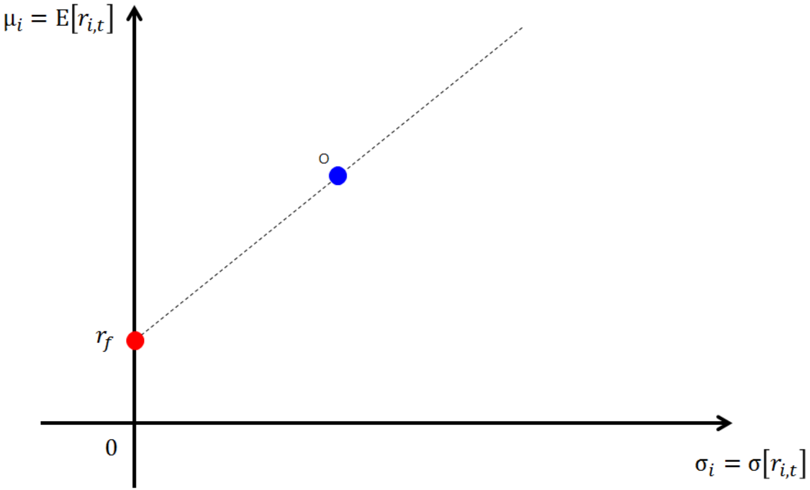

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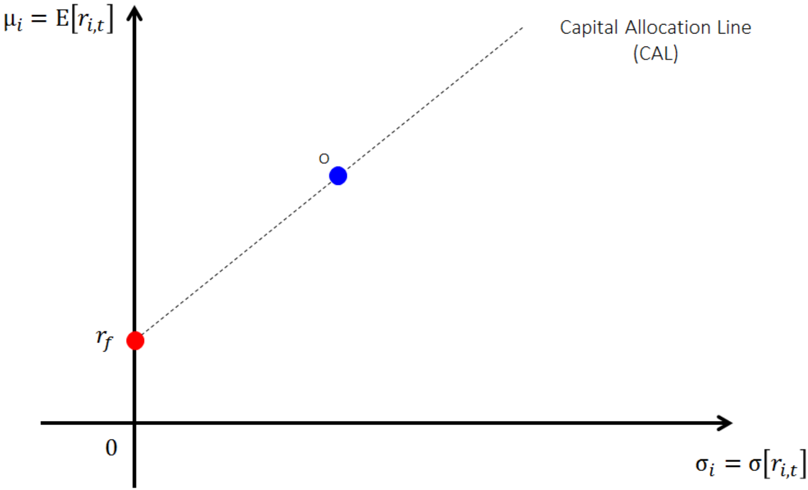
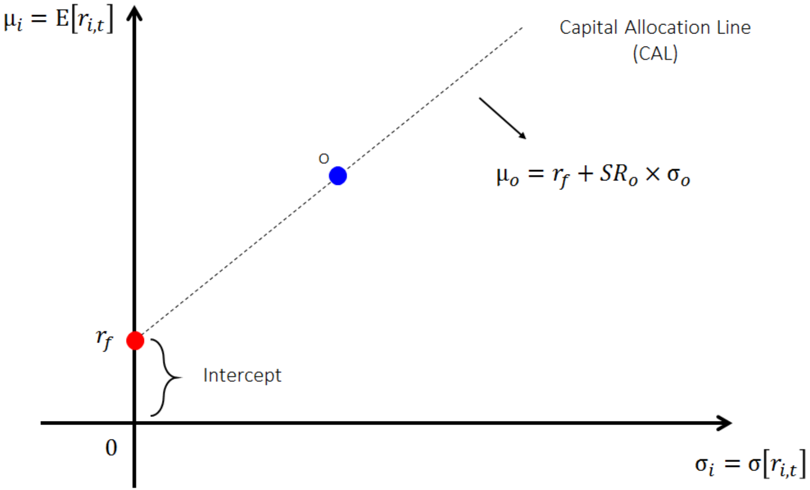


$\sigma[r_t] \times \mathbb{E}[r_t]$: Sharpe Ratio



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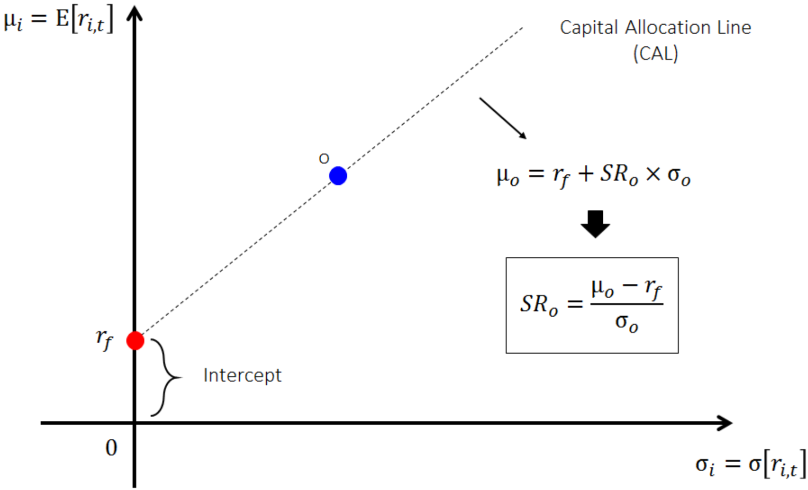
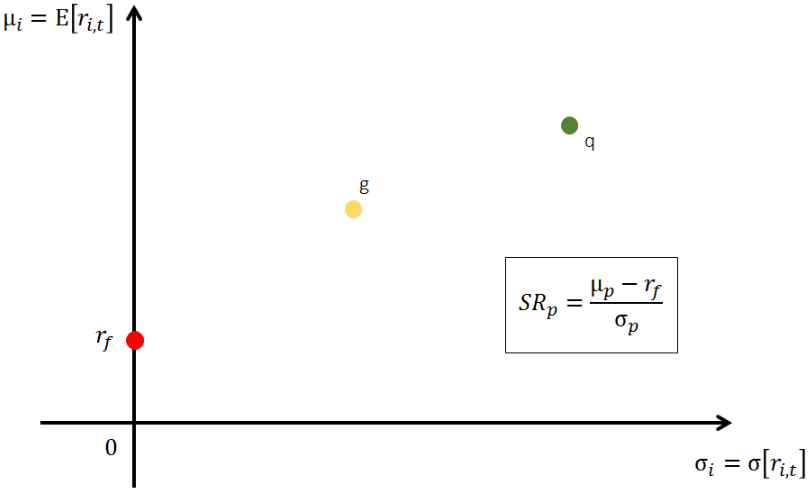


$\sigma[r_t] \times \mathbb{E}[r_t]$: Portfolio Comparison using Sharpe Ratio



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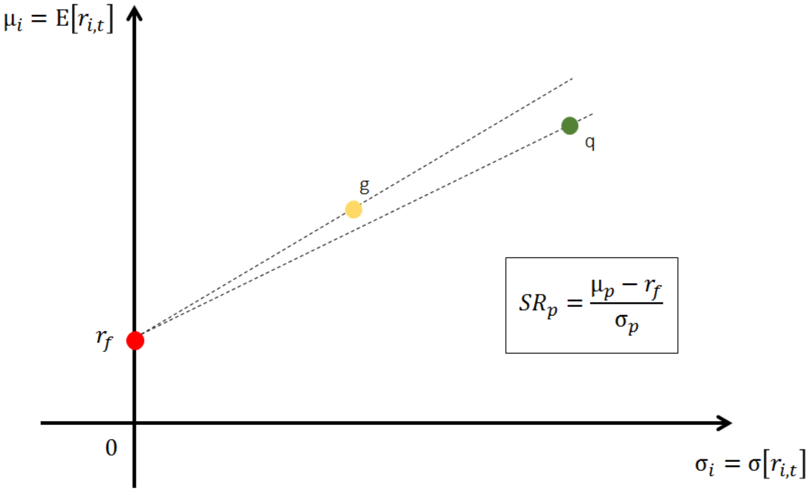

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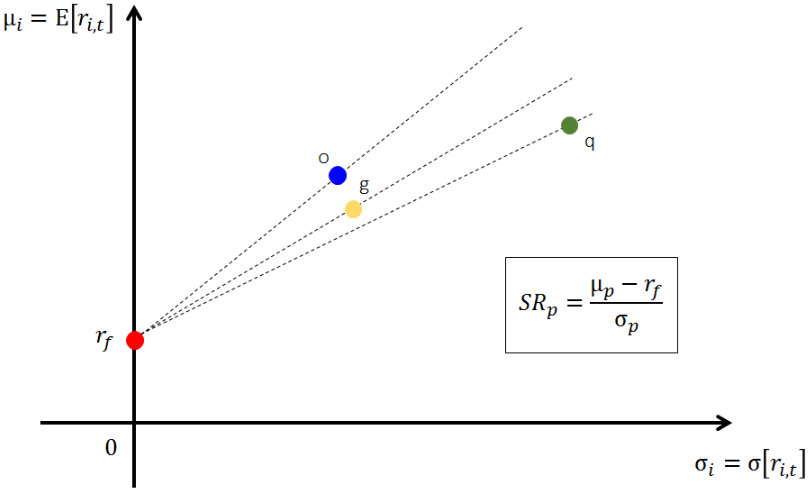
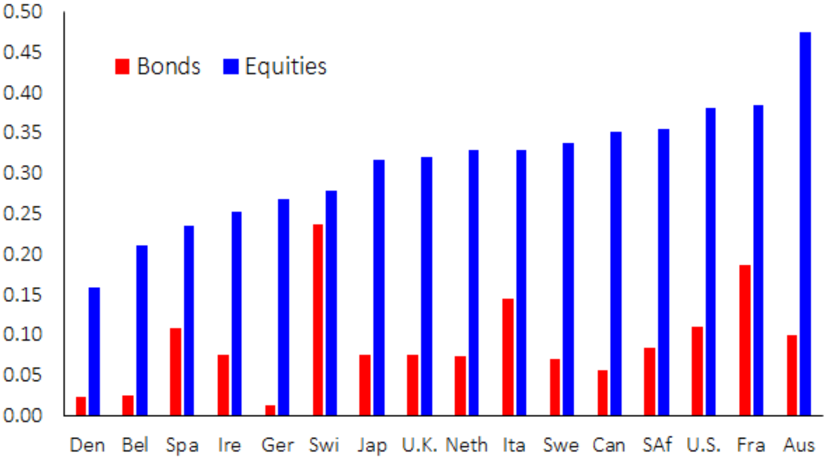
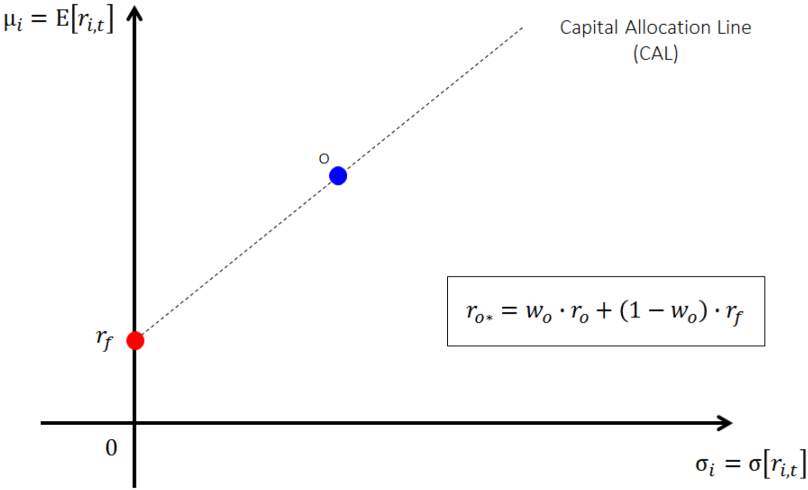


Sharpe Ratios around the World from 1900-2000



Source: Dimson et al (2002) - *Triumph of the optimists: 101 years of global investment returns*

$\sigma[r_t] \times \mathbb{E}[r_t]$: Complete Portfolio o^*



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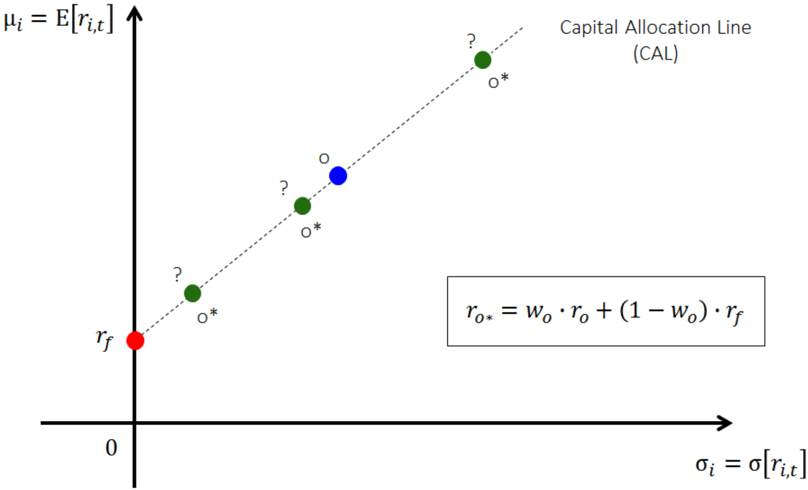

Properties of $\mathbb{E}[r_{o^*}]$ and $\sigma[r_{o^*}]$

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Deciding on Complete Portfolio o^*

- Investors happiness depends on $\sigma_{o^*} = \sigma\left[r_{o^*}\right]$ and $\mu_{o^*} = \mathbb{E}\left[r_{o^*}\right]$

- We can model investors happiness (called utility function) as:

$$U(\sigma_{o^*}, \mu_{o^*}) = \mu_{o^*} - 0.5 \cdot A \cdot \sigma_{o^*}^2$$

$$= \underbrace{w_o \cdot \mu_o + (1 - w_o)\, r_f}_{\mu_{o^*}} - 0.5 \cdot A \cdot \underbrace{w_o^2 \cdot \sigma_o^2}_{\sigma_{o^*}^2}$$

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- Investors happiness depends on $\sigma_{o^*} = \sigma\left[r_{o^*}\right]$ and $\mu_{o^*} = \mathbb{E}\left[r_{o^*}\right]$

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U\left(\sigma_{o^*}, \mu_{o^*}\right) &= \mu_{o^*} - 0.5 \cdot A \cdot \sigma_{o^*}^2 \\
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The task of forming the "complete portfolio" (called o^*), is separated into two different tasks: (i) finding the "optimal" risky portfolio (called o) and (ii) getting o^* by combining portfolio o with r_f. Which of the following is true regarding this process:

a) Two investors using the same inputs to step (i) always end up with the same risky portfolio, o

b) Step (ii) is investor specific. However, if two investors have the same risk aversion, they always end up with the same o^*

c) Portfolio o can only be found if we first find the entire efficient frontier

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Outline

Overview

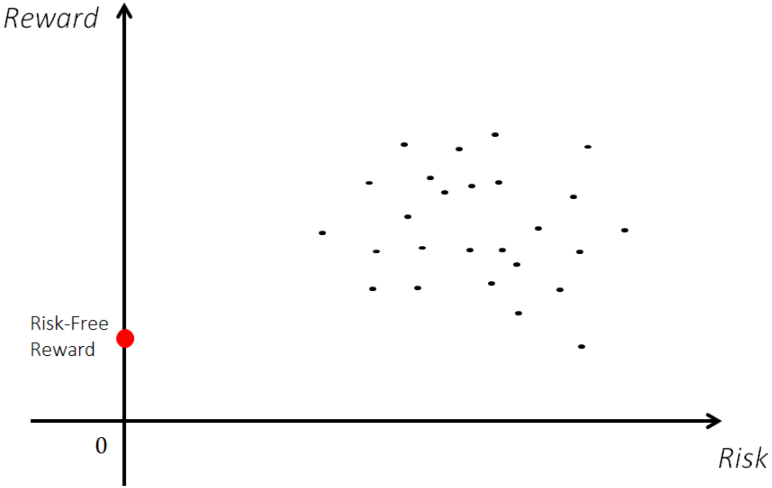
The Statistics of Security Returns

The Efficient Frontier

Capital Allocation Line

Index Models

This Section: Inputs to $\sigma[r_t] \times \mathbb{E}[r_t]$ Framework



Estimating Covariances

- Use each data observation as a "scenario" with equal probability:

$$Cov\left[r_A, r_B\right] = \sum_s p\left(s\right) \times \left\{r_A\left(s\right) - \mathbb{E}\left[r_A\right]\right\} \times \left\{r_B\left(s\right) - \mathbb{E}\left[r_B\right]\right\}$$

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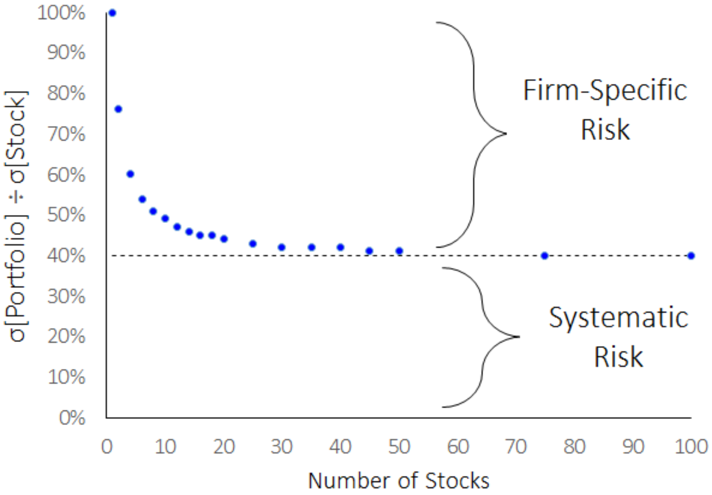

Source: "Statman (1987) - *How many stocks make a diversified portfolio*"

Index Model: Structure

- Decomposing returns into two components:

$$r_{i,t} - r_f = \beta_i \cdot \underbrace{(r_{M,t} - r_f)}_{systematic} + \underbrace{\alpha_i + e_{i,t}}_{firm-specific}$$

$$\Downarrow$$

$$\mathbb{E}\left[r_{i,t}\right] = r_f + \alpha_i + \beta_i \cdot \left(\mathbb{E}\left[r_{M,t}\right] - r_f\right)$$

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Index Model: Reduction in Number of Estimates

- With an index model with N assets, we need:
  - N estimates of α_i, β_i and $\sigma\left[e_{i,t}\right]$
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- The reduction in the number of estimates is substantial:
  - $N = 50$ securities $\Longrightarrow$ 1,355 estimates
  - $N = 50$ securities $\Longrightarrow$ 152 estimates (with index model)

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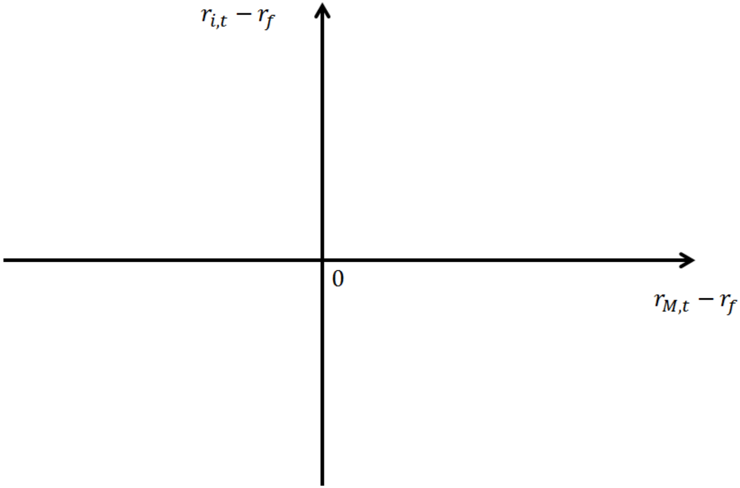

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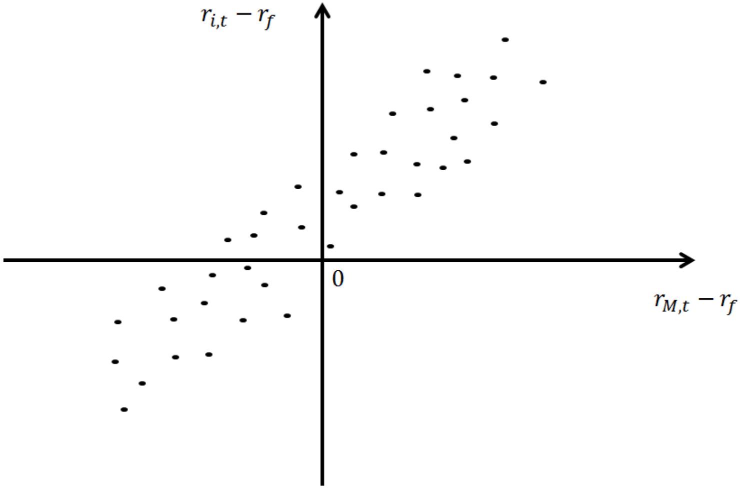

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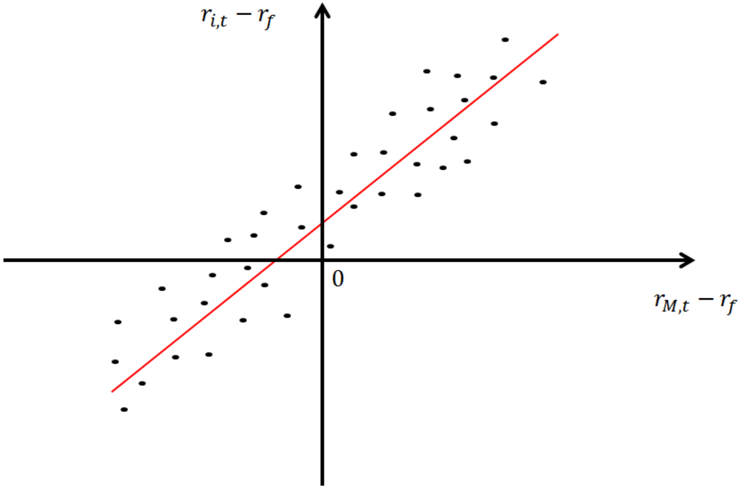

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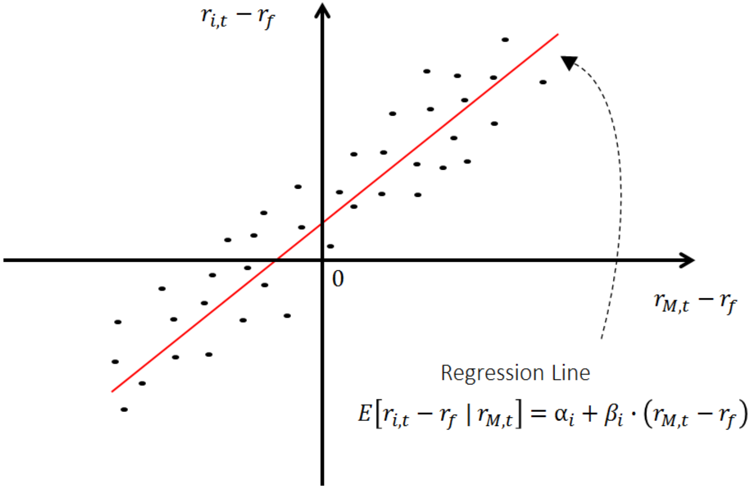
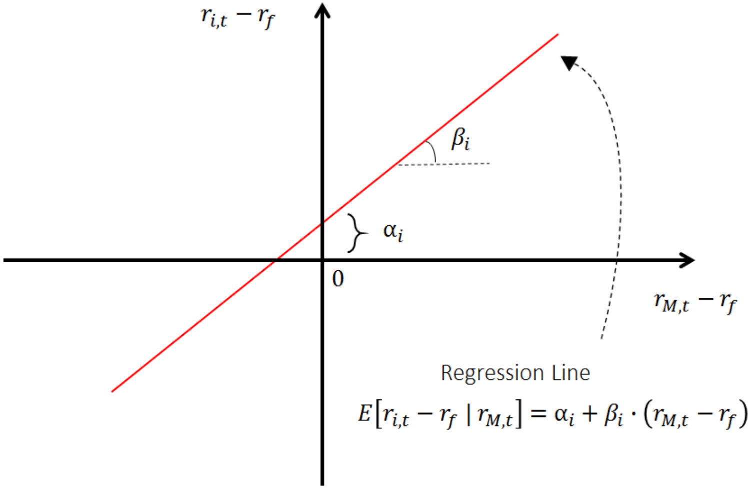
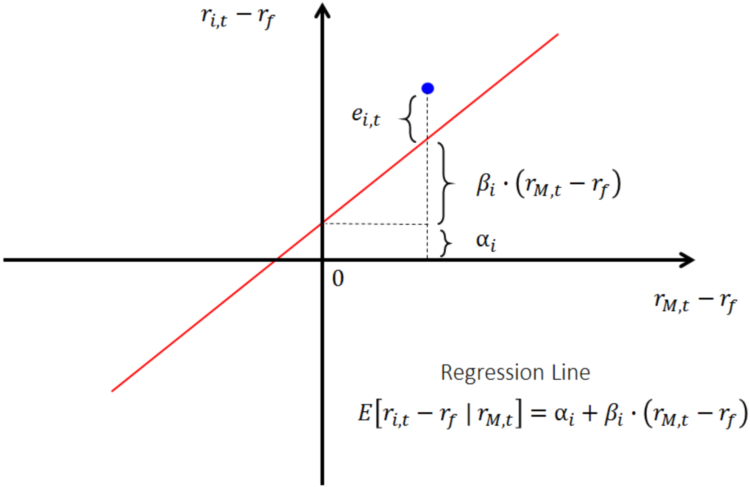


Index Model as a Regression: α_i and β_i



Index Model as a Regression: Decomposing $r_{i,t}$



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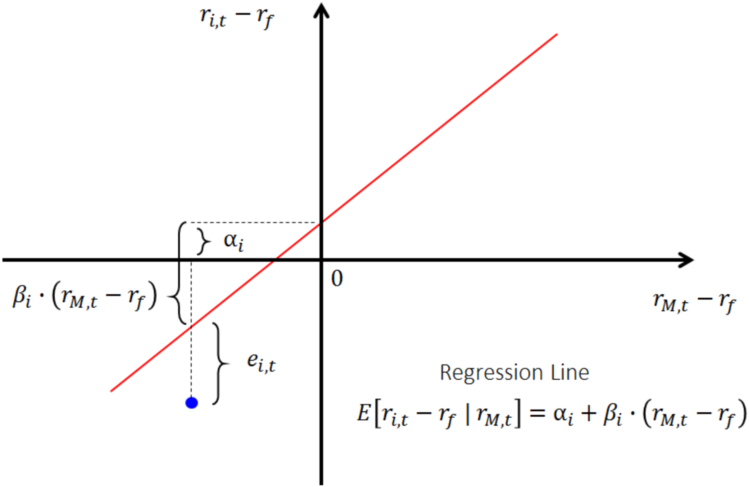

Index Model: Systematic × Firm-Specific Risk

- Decomposing risk into two components:

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$$= \alpha_p + \beta_p \cdot (r_M - r_f) + \left(\frac{1}{N}\sum_i e_i\right)$$

$$\cong \alpha_p + \beta_p \cdot (r_M - r_f)$$

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$$r_p - r_f = \underbrace{\left(\frac{1}{N}\sum_i \alpha_i\right)}_{\alpha_p} + \underbrace{\left(\frac{1}{N}\sum_i \beta_i\right)}_{\beta_p} \cdot (r_M - r_f) + \left(\frac{1}{N}\sum_i e_i\right)$$

$$= \alpha_p + \beta_p \cdot (r_M - r_f) + \left(\frac{1}{N}\sum_i e_i\right)$$

$$\cong \alpha_p + \beta_p \cdot (r_M - r_f)$$

$$\Downarrow$$

$$\sigma^2[r_p] \cong \beta_p^2 \cdot \sigma^2[r_M]$$

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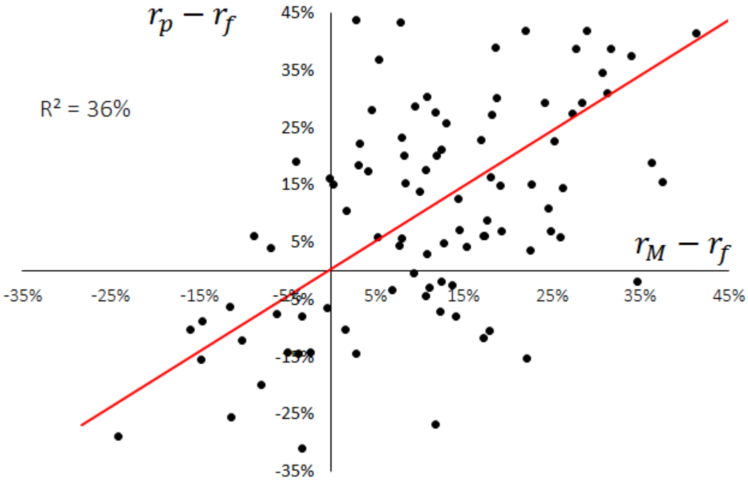
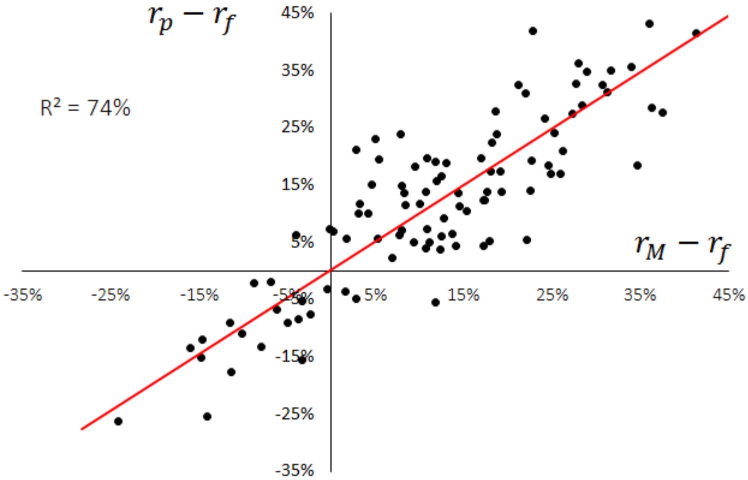
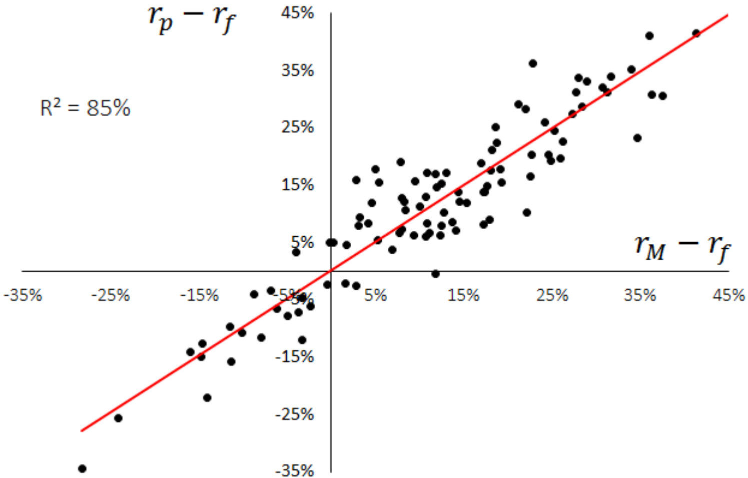


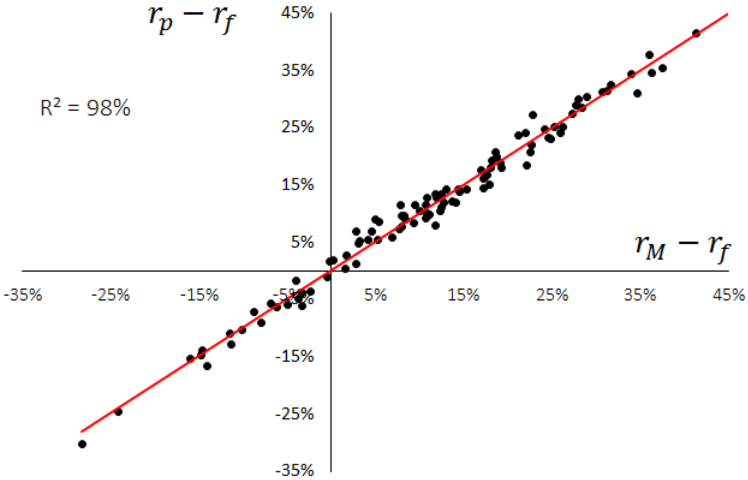
$N = 5$



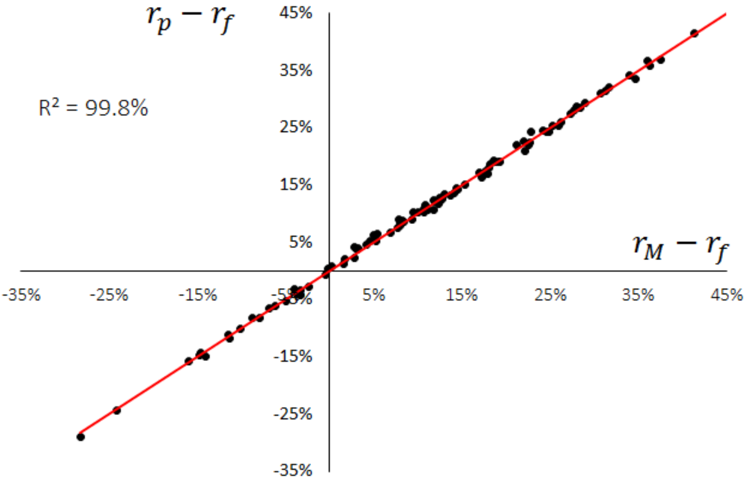
$N = 10$



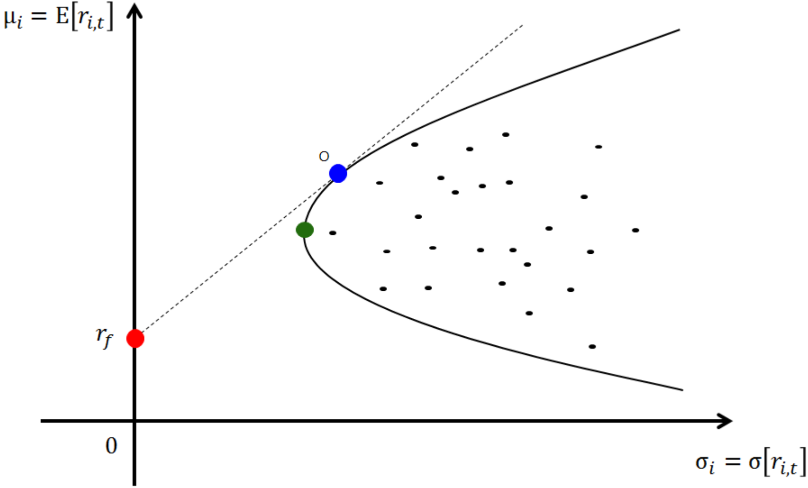
$N = 100$



$N = 1,000$



Index Model: Finding the Tangency Portfolio



Index Model: Finding the Tangency Portfolio

- It turns out that (using the index model):

$$r_o = \underbrace{w_A \cdot r_A}_{Active} + \underbrace{(1 - w_A) \cdot r_M}_{Passive}$$

with

$$w_A^0 = \frac{\alpha_A / \sigma^2[e_A]}{(\mathbb{E}[r_M] - r_f) / \sigma_M^2} \quad and \quad w_A = \frac{w_A^0}{1 + w_A^0 \cdot (1 - \beta_A)}$$

- Moreover, weight of asset i in the active portfolio is:

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You are considering using an index model when estimating inputs for your portfolio optimization problem. All of the followings are advantages of this approach of estimating inputs, except:

a) It reduces substantially the number of parameters to be estimated

b) It creates a clear decomposition between systematic and firm-specific risk

c) It breaks the optimal risky portfolio into a passive portfolio and an active position, which allows you to understand how you are deviating from the given index

d) It provides you with estimates that rely on a lower number of assumptions about returns

e) It provides a simple way to incorporate the fact that many events in the market affect several assets simultaneously

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A Word of Caution Regarding Portfolio Theory

- Portfolio theory provides you with an extremely useful tool for portfolio formation. However:
  - "Garbage in, garbage out" principle
  - Portfolio Theory is silent about how to estimate $\mathbb{E}[r_t]$ and $\sigma[r_t]$. Forward looking estimates are key (security analysis)
  - Maximum Sharpe Ratio portfolio is very sensitive to model inputs (especially $\mathbb{E}[r_t]$)
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