Module 3: Factor Models

(BUSFIN 4221 - Investments)

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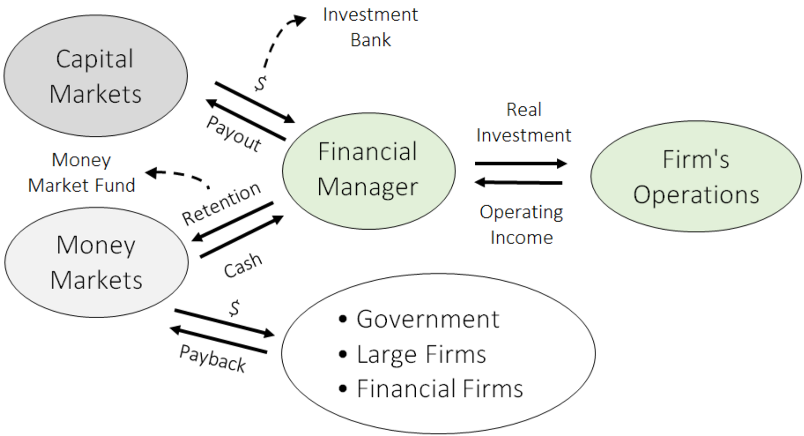
The Ohio State University

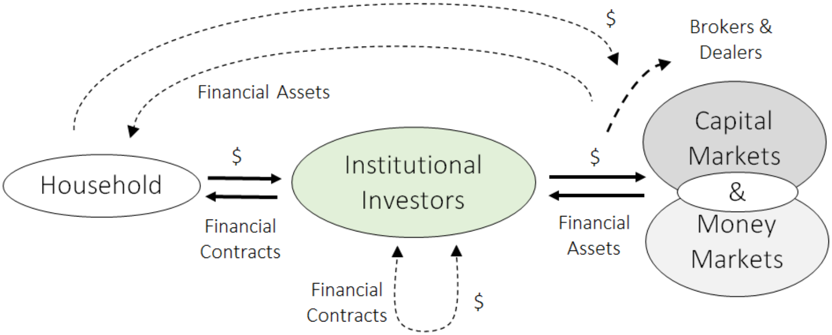
Fall 2016

Outline

Module 1 - The Demand for Capital



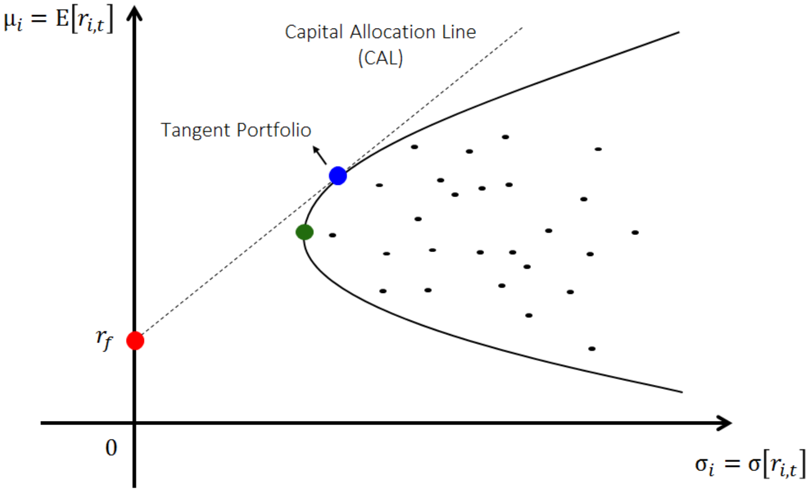
Module 1 - The Supply of Capital



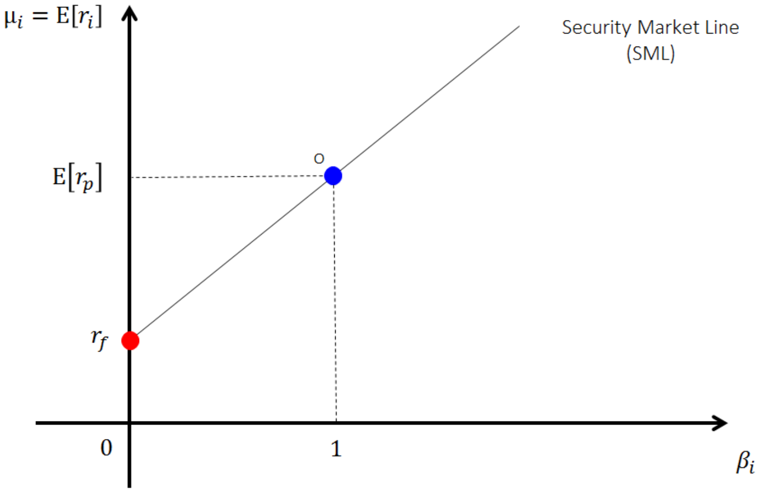
Module 1 - Investment Principle

$$PV_t = \sum_{h=1}^{\infty} \frac{\mathbb{E}_t\left[CF_{t+h}\right]}{\left(1 + dr_{t,h}\right)^h}$$

Module 2 - Portfolio Theory

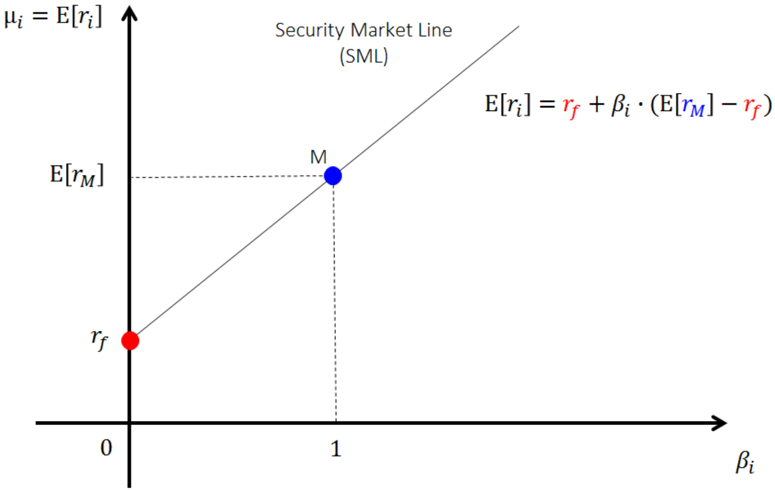


This Module: Factor Models

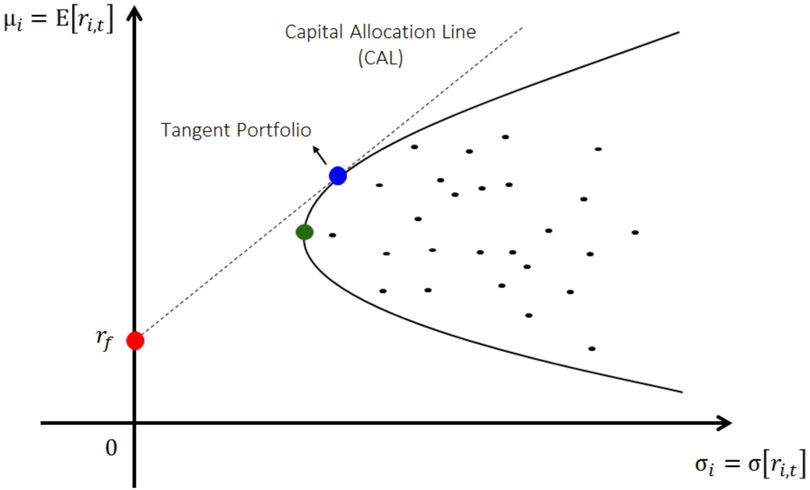


Outline

This Section: CAPM



Capital Allocation Line



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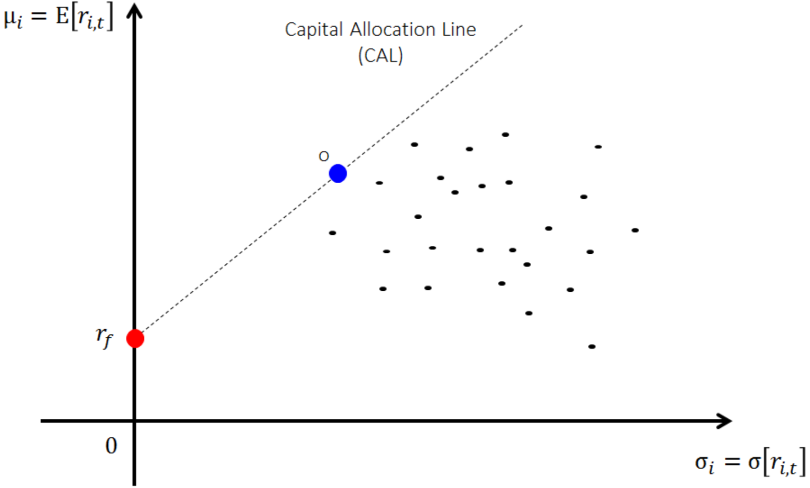

Asset i Contribution to Portfolio Reward and Risk

$$\mathbb{E}[r_o] - r_f = \mathbb{E}[w_1 \cdot r_1 + ... + w_N \cdot r_N] - r_f$$

$$= \underbrace{w_1 \cdot (\mathbb{E}[r_1] - r_f)}_{Contribution\ of\ Asset\ 1} + ... + \underbrace{w_N \cdot (\mathbb{E}[r_N] - r_f)}_{Contribution\ of\ Asset\ N}$$

$$\sigma^2[r_o] = Cov[r_o, r_o]$$

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- Since investors want to maximize $(\mathbb{E}[r_o]-r_f)/\sigma^2[r_o]$, the contribution of each asset to this expression must be the same:

$$\frac{\mathbb{E}[r_o]-r_f}{\sigma^2[r_o]} = \frac{w_i \cdot (\mathbb{E}[r_i]-r_f)}{w_i \cdot Cov[r_i, r_o]}$$

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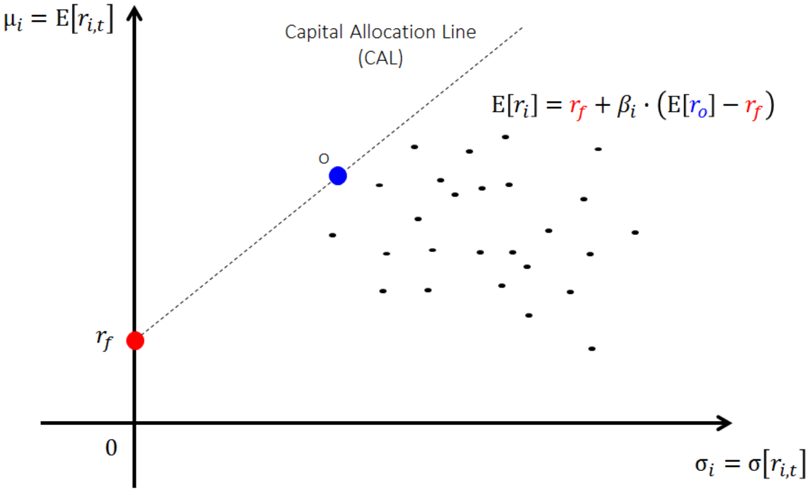
CAPM: The Argument



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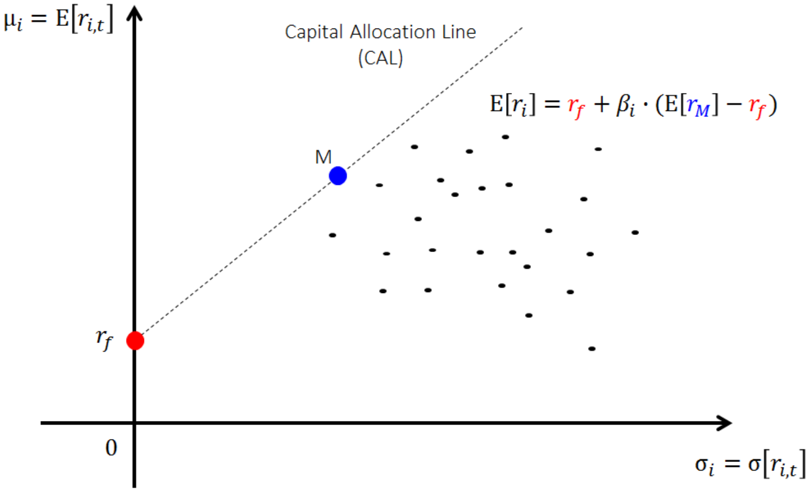

CAPM: Understanding the key Prediction

$$\mathbb{E}\left[r_i\right] = \underbrace{r_f}_{Risk-Free\ Reward} + \underbrace{\beta_i}_{Risk\ Exposure} \cdot \underbrace{\left(\mathbb{E}\left[r_M\right] - r_f\right)}_{Risk\ Premium}$$

- $\beta_i = {}^{Cov(r_i, r_M)}/_{\sigma^2[r_M]}$
- $\uparrow \beta_i \implies \uparrow \mathbb{E}\left[r_i\right]$:
- $\sigma^2\left[r\right]$ is not the right measure of risk. β is
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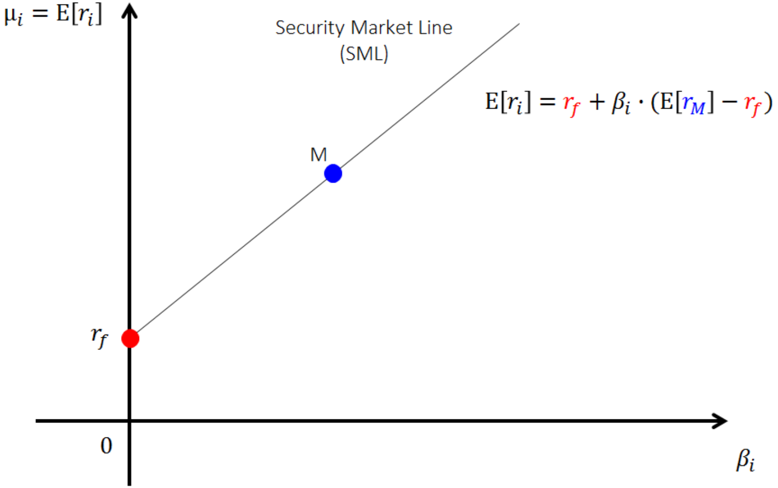

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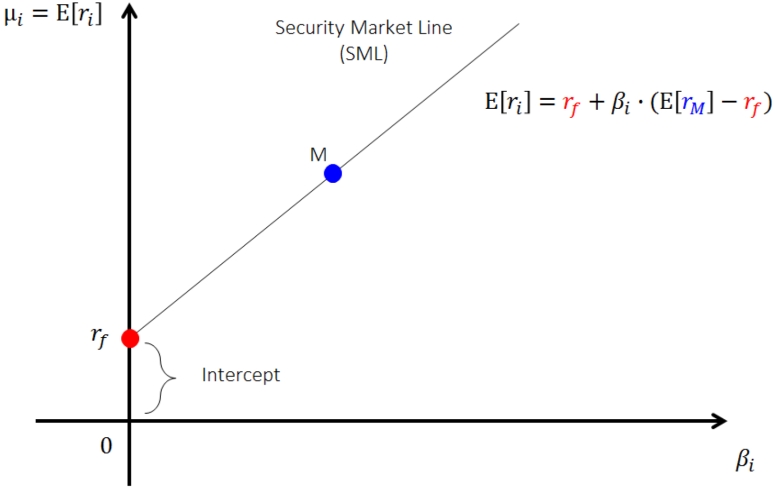

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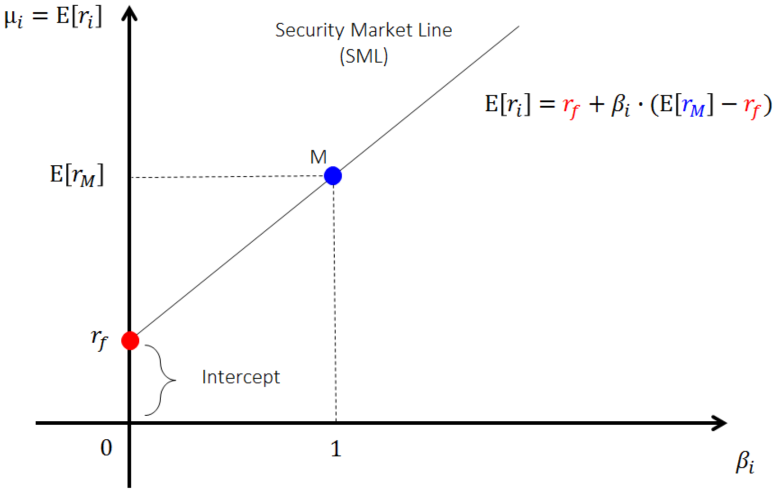

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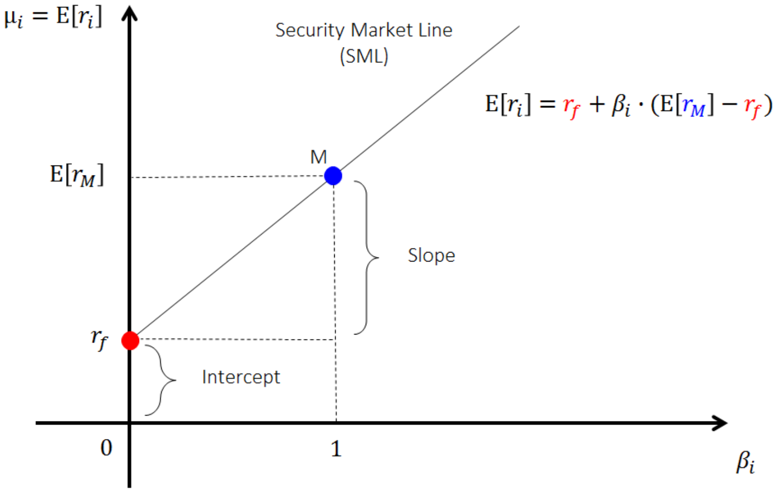

CAPM: The Risk-Premium

$$\mathbb{E}[r_i] = \underbrace{r_f}_{Risk-Free\ Reward} + \underbrace{\beta_i}_{Risk\ Exposure} \cdot \underbrace{(\mathbb{E}[r_M] - r_f)}_{Risk\ Premium}$$

- When investors decide on their complete portfolios by maximizing their "Happiness" then: $\mathbb{E}[r_M] - r_f = A \cdot \sigma_M^2$
- This means that the Risk Premium:
  - Increases with market volatility:

    $\uparrow \sigma_M \quad \Rightarrow \quad \uparrow \mathbb{E}[r_M] - r_f$
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Suppose the CAPM holds: $\mathbb{E}[r_i] = r_f + \beta_i \cdot (\mathbb{E}[r_M] - r_f)$. Consider two stocks, A and B, with $\mathbb{E}[r_A] > \mathbb{E}[r_B]$ at time t. Between t and $t+1$ there is an increase in the volatility of the market portfolio, which induces an increase the market Risk-Premium (nothing else changes). What can you say about the expected return gap $\mathbb{E}[r_A] - \mathbb{E}[r_B]$?

a) It will increase from t to $t+1$

b) It will decrease from t to $t+1$.

c) It will remain the same from t to $t+1$ (β's did not change).

d) It will revert (become negative) from t to $t+1$ since stock A will be hit harder.

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CAPM: Relation to Index Model

$$\text{Index Model}: \quad r_{i,t} - r_f = \alpha_i + \beta_i \cdot (r_{M,t} - r_f) + e_{i,t}$$

and

$$\text{CAPM}: \quad \mathbb{E}[r_i] - r_f = \beta_i \cdot (\mathbb{E}[r_M] - r_f)$$

$$\Downarrow$$

$$\alpha_i = 0$$

- The usual systematic x firm-specific risk decomposition holds:

$$\sigma^2[r_i] = \beta_i^2 \cdot \sigma^2[r_M] + \sigma^2[e_i]$$

- We can estimate β using as the slope of a regression with $y = r_{i,t} - r_f$ and $x = r_{M,t} - r_f$

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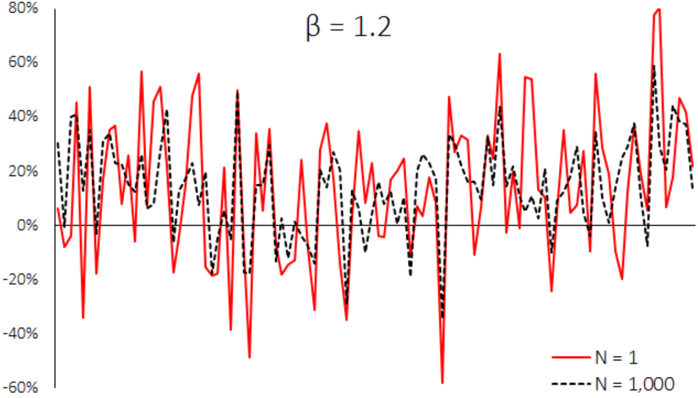
CAPM: β Effect on Portfolio Volatility



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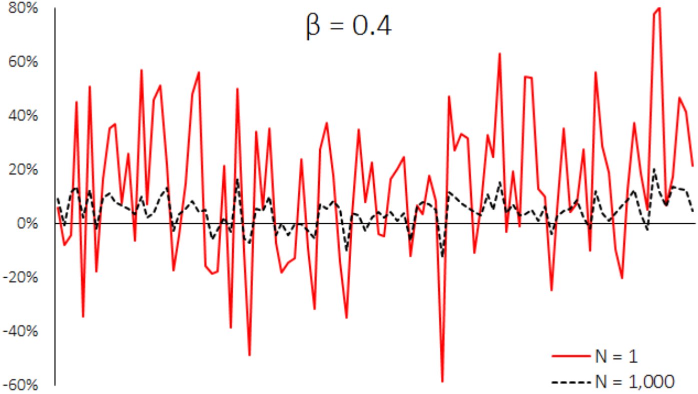

CAPM: Applications

- There are several CAPM applications, but two are particularly important since they are often used by market participants

- Portfolio Management: Risk Adjusted Returns

$$\mathbb{E}[r_i] - r_f = \alpha_i + \beta_i \cdot (\mathbb{E}[r_M] - r_f)$$

- You can use the simplest approach:

$$\widehat{\alpha}_i = \overline{r_i - r_f} - \widehat{\beta}_i \cdot \overline{r_M - r_f}$$

- Or you can use sophisticated security analysis:

$$\widehat{\alpha}_i = \widehat{\mathbb{E}}[r_i - r_f] - \widehat{\beta}_i \cdot \widehat{\mathbb{E}}[r_M - r_f]$$

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- There are several CAPM applications, but two are particularly important since they are often used by market participants

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$$PV_t = \sum_{h=1}^{T} \frac{\mathbb{E}_t\left[CF_{t+h}\right]}{\left(1+dr_{t,h}\right)^h}$$

- The CAPM provides an estimate for the discount rate:

$$\begin{aligned}
\left(1+dr_{t,h}\right)^h &= \left(1+\widehat{\mathbb{E}}\left[r_i\right]\right)^h \\
&= \left(1+r_f+\widehat{\beta}_i\cdot\widehat{\mathbb{E}}\left[r_M-r_f\right]\right)^h \\
&\cong \left(1+r_f+\widehat{\beta}_i\cdot 6\%\right)^h
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CAPM: The Paradox

- For the CAPM to hold, we need active investors in the market (using Portfolio Theory)
- But the main prediction of the CAPM is that the market portfolio is efficient
- Most people can easily buy an ETF that mimics the (stock) market portfolio
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- Complete Agreement (or "homogeneous expectations")!
- No private information!
- No taxes!
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Use the CAPM in a Sensible way

- There are too many (false) assumptions for the CAPM to hold
- The question you should ask is not whether the CAPM holds or not. Instead you should ask: when is it reasonable to use it?
- As any model in Finance, when you blindly apply the CAPM you might face serious issues: mismeasure risk, mismeasure expected return, invest in unreasonable projects...
- Researchers have been working on improving the CAPM over more than three decades. Much progress has been made.
- Yet, most models keep the key insight from the CAPM: systematic risk matters!

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Which of the following statements is false regarding the CAPM?

a) The CAPM links $\mathbb{E}[r_i]$ and systematic risk, measured by $\beta_i = {}^{Cov[r_i, r_M]}/_{\sigma^2[r_M]}$, directly. It says that asset A has higher expected return than asset B if and only if $\beta_A > \beta_B$

b) It is possible to have a world in which all investors use Portfolio Theory to decide on their portfolios and, yet, the CAPM predictions are false

c) Within the CAPM, the best risk measure for an asset or portfolio is its volatility: $\sigma^2[r]$

d) The CAPM prediction for $\mathbb{E}[r_i]$ can be used as a discount rate to be applied in Net Present Value (NPV) applications

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Outline

This Section: APT

"No free lunch" rule in Wall Street (+ index model) implies:

$$\mathbb{E}[r_p] = r_f + \beta_p \cdot (\mathbb{E}[r_M] - r_f)$$

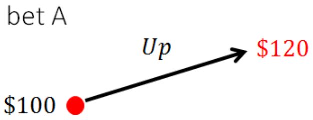
for any "well diversified" portfolio p

No Abitrage Principle

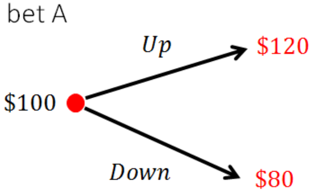
bet A

$100

No Abitrage Principle



No Abitrage Principle



No Abitrage Principle

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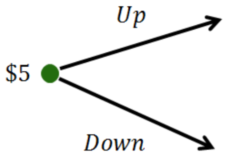
Take bet A
&
Offer bet B



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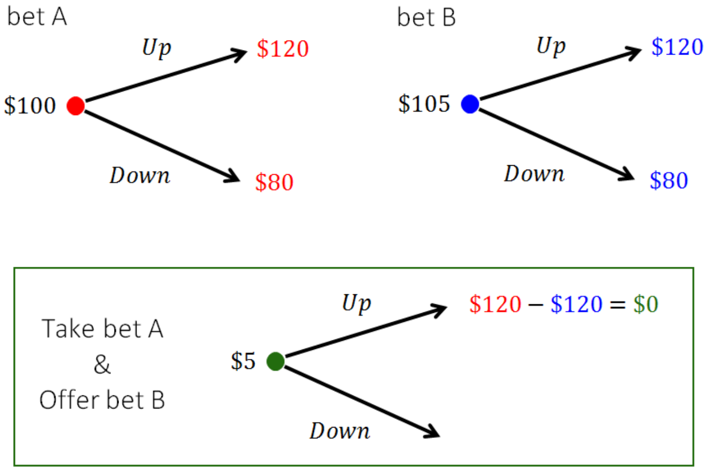

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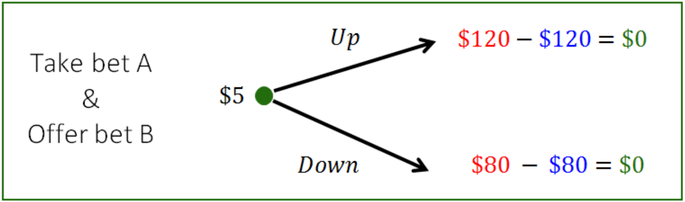

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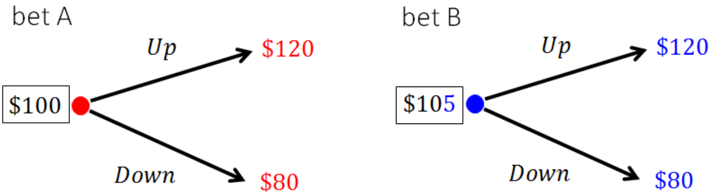

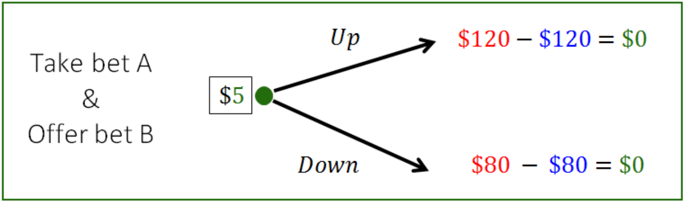

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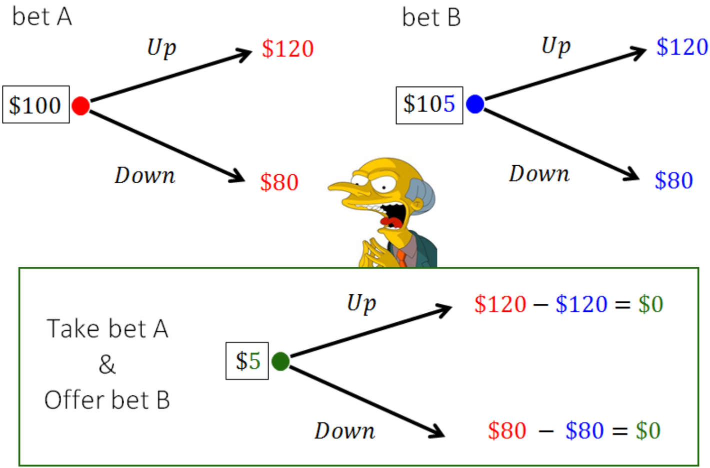

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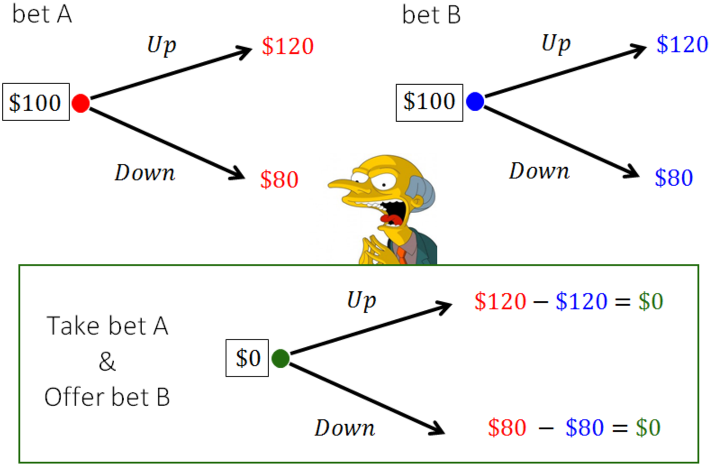
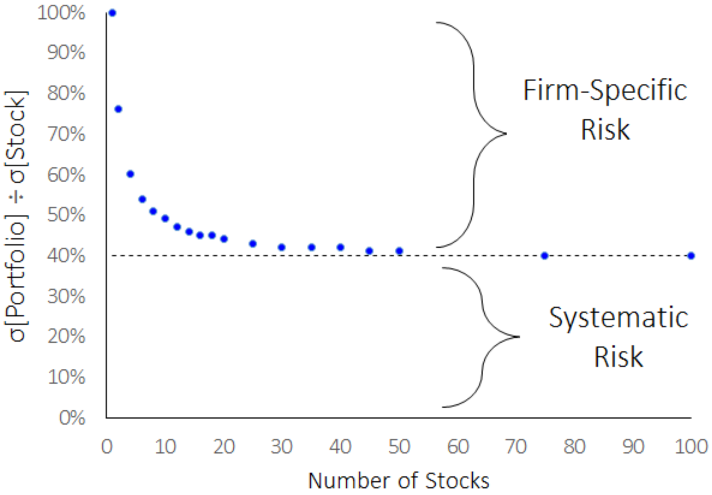


Systematic × Firm-Specific Risk



Source: "Statman (1987) - *How many stocks make a diversified portfolio*"

Systematic × Firm-Specific Risk

- Assume an index model holds:

$$r_{i,t} - r_f = \alpha_i + \beta_i \cdot (r_{M,t} - r_f) + e_{i,t}$$

$$\Downarrow$$

$$\sigma^2 [r_{i,t}] = \underbrace{\beta_i^2 \cdot \sigma^2 [r_{M,t}]}_{systematic\ risk} + \underbrace{\sigma^2 [e_{i,t}]}_{firm-specific\ risk}$$

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Firm-Specific Risk Vanishes in Well Diversified Portfolios

- When we form a (equal-weighted) portfolio $r_p = \frac{1}{N}\sum_{i=1}^{N} r_i$:

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$$= \alpha_p + \beta_p \cdot (r_M - r_f) + \left(\frac{1}{N}\sum_i e_i\right)$$

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$$r_p - r_f \cong \alpha_p + \beta_p \cdot (r_M - r_f)$$

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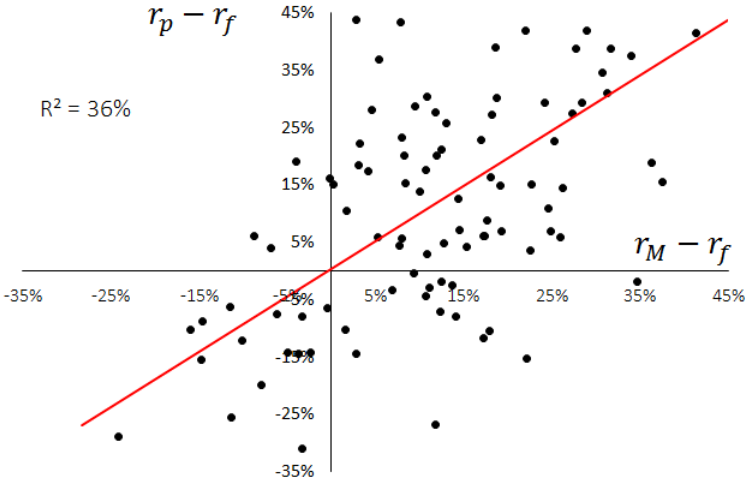
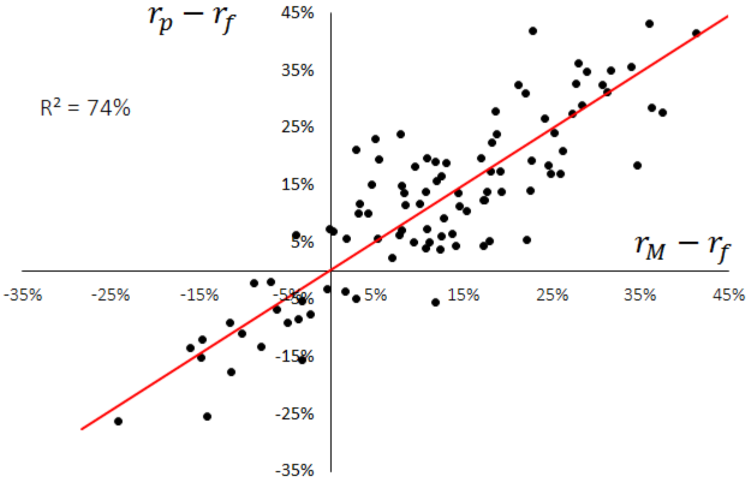
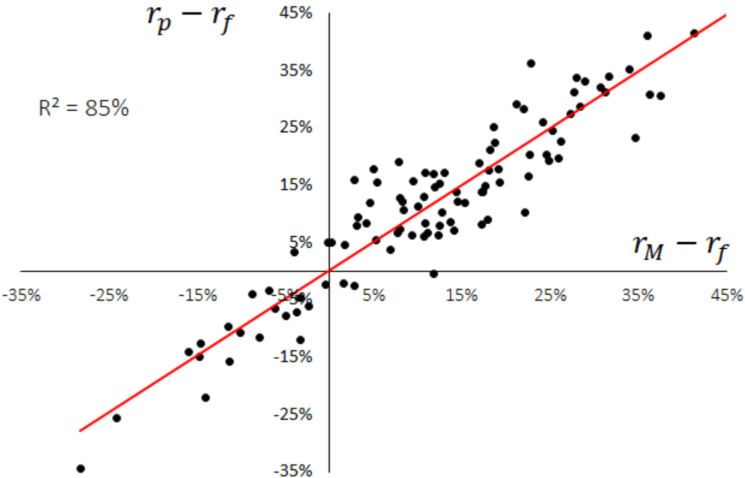


$N = 5$



$N = 10$



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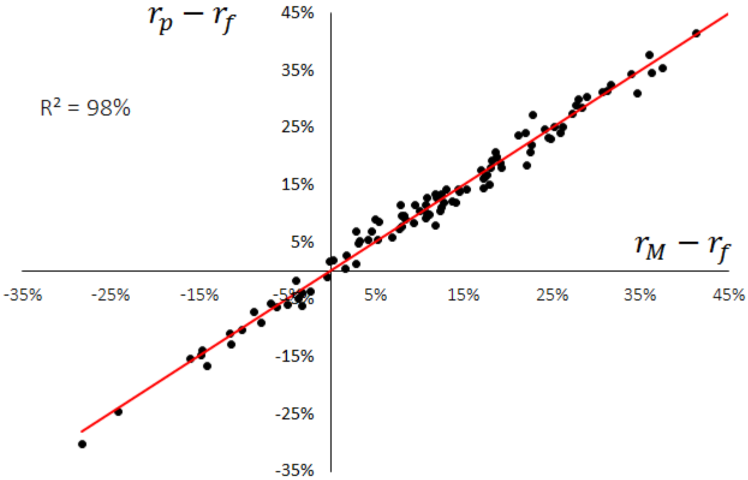
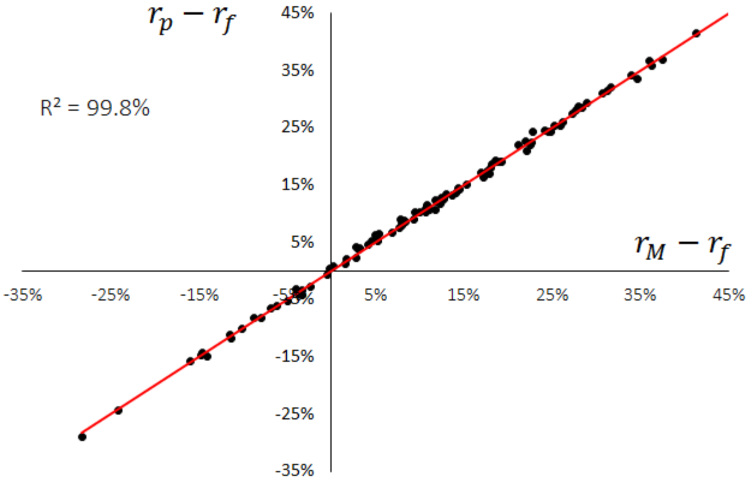


$N = 1,000$



Creating an Arbitrage Strategy

- For a well diversified portfolio, p, we have:

$$r_p - r_f = \alpha_p + \beta_p \cdot (r_M - r_f)$$

$$r_M - r_f = 0 + 1 \cdot (r_M - r_f)$$

- Let's create a portfolio, $r_z = w_p \cdot r_p + (1 - w_p) \cdot r_M$, with zero systematic risk:

$$\beta_z = w_p \cdot \beta_p + (1 - w_p) \cdot 1 = 0$$

$$\Downarrow$$

$$w_p = \frac{1}{1 - \beta_p}$$

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Creating an Arbitrage Strategy

- For a well diversified portfolio, p, we have:

$$r_p - r_f = \alpha_p + \beta_p \cdot (r_M - r_f)$$
$$r_M - r_f = 0 + 1 \cdot (r_M - r_f)$$

- Let's create a portfolio, $r_z = w_p \cdot r_p + (1 - w_p) \cdot r_M$, with zero systematic risk:

$$\beta_z = w_p \cdot \beta_p + (1 - w_p) \cdot 1 = 0$$
$$\Downarrow$$
$$w_p = \frac{1}{1 - \beta_p}$$

Creating an Arbitrage Strategy

- Our portfolio, r_z, has $\beta_z = 0$ and $\alpha_z = w_p \cdot \alpha_p$. Hence:

$$
\begin{aligned}
r_z - r_f &= \alpha_z + \beta_z \cdot (r_M - r_f) \\
&= \alpha_z \\
&\Downarrow \\
r_z &= r_f + \alpha_z
\end{aligned}
$$

- We just created a risk-free asset paying an interest rate higher than the risk-free rate (lower if $\alpha_z < 0$)

- This cannot be sustainable (smart investors will arbitrage that difference away by taking contrary positions on the r_f and r_z)

- As a consequence, α_z is driven to zero. Of course, α_z depends only on α_p, which means that $\alpha_p = 0$

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APT x CAPM

- CAPM (Equilibrium Principle):

$$\textit{Portfolio Theory}$$
$$\Downarrow$$
$$\mathbb{E}[r_i] = r_f + \beta_i \cdot (\mathbb{E}[r_M] - r_f)$$

- APT (Non-Arbitrage Principle):

$$r_{i,t} - r_f = \alpha_i + \beta_i \cdot (r_{M,t} - r_f) + e_{i,t}$$
$$\Downarrow$$
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In principle, which of the following assumptions is necessary for the APT to work?

a) Many investors need to try to take advantage of arbitrage opportunities offered by deviations from the APT implications.

b) Investors must prefer lower risk, $\sigma^2[r]$, and higher reward, $\mathbb{E}[r]$.

c) Investors need to be homogeneous in the sense that they estimate $\sigma^2[r]$ and $\mathbb{E}[r]$ the same way.

d) At least one investor must have access to lending/borrowing at the risk-free and use this capacity to take advantage of arbitrage opportunities offered by deviations from the APT implications.

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Outline

Overview

Capital Asset Pricing Model

Arbitrage Pricing Theory

Multifactor Models

Empirical Evidence

This Section: Multiple Factors

We can generalize the logic in the previous sections:

$$\mathbb{E}\left[r_i\right] = r_f + \beta_i \cdot \left(\mathbb{E}\left[r_M\right] - r_f\right)$$
$$+ \beta_{i,A} \cdot \mathbb{E}\left[r_A - r_a\right]$$
$$+ \beta_{i,B} \cdot \mathbb{E}\left[r_B - r_b\right]$$
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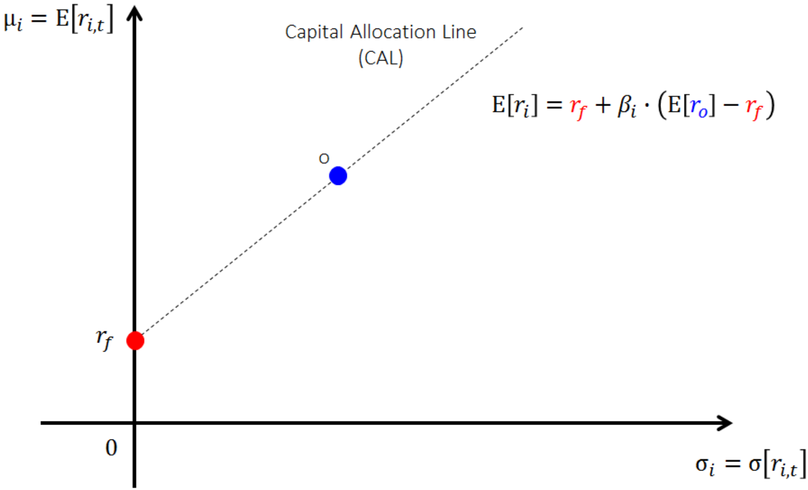
CAPM: Multifactor Equilibrium Models



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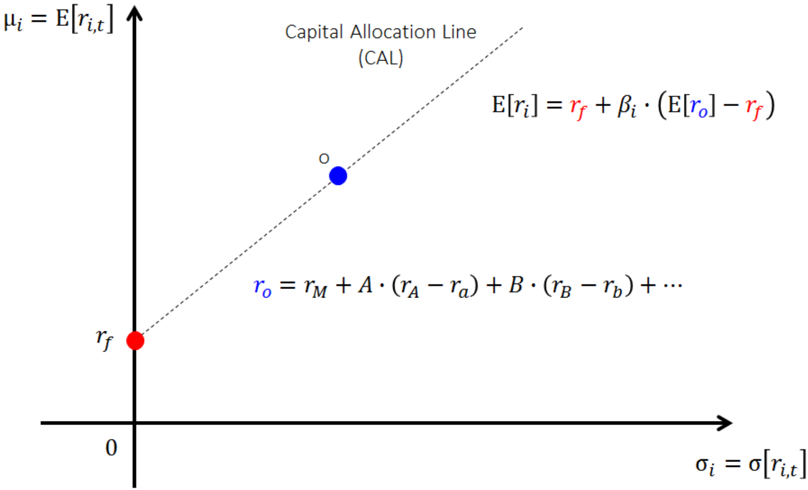

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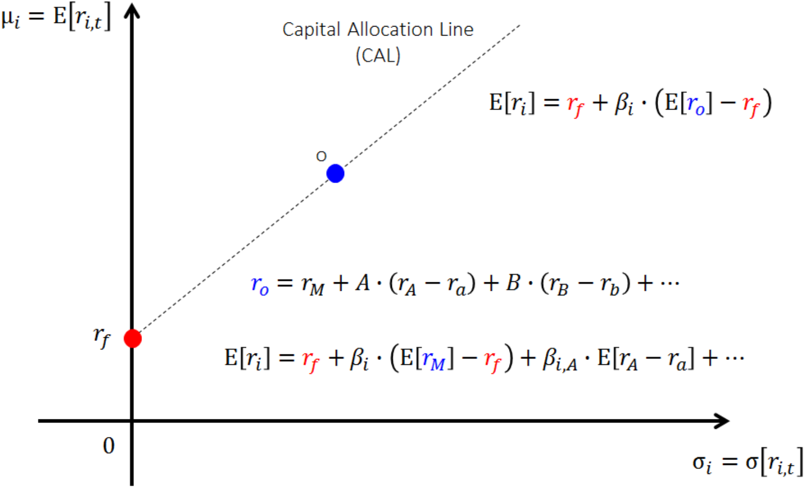

APT: Multifactor Arbitrage Pricing Models

$$r_{i,t} - r_f = \alpha_i + \beta_i \cdot (r_{M,t} - r_f) + \beta_{i,A} \cdot (r_A - r_a) + ... + e_{i,t}$$

$$+$$

No free lunch in Wall street

$$\Downarrow$$

$$\mathbb{E}[r_p] = r_f + \beta_p \cdot (\mathbb{E}[r_M] - r_f) + \beta_{p,A} \cdot \mathbb{E}[r_A - r_a] + ...$$

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That is, for any "well diversified" portfolio: $\alpha_p = 0$

Outline

This Section: CAPM Tests and the 3 Factor Model

CAPM empirical tests indicate it is an inadequate model. However, its logic is relevant and the most commonly applied factor model was created from a careful analysis of the CAPM failure.

CAPM: SML Prediction

- From CAPM, we have:

$$\mathbb{E}[r_i] = r_f + \beta_i \cdot (\mathbb{E}[r_M] - r_f)$$

$$\overline{r}_i = \lambda_0 + \widehat{\beta}_i \cdot \lambda_1 + \epsilon_i$$

$$\Downarrow$$

$$\lambda_0 = r_f \quad \textit{and} \quad \lambda_1 = \mathbb{E}[r_M] - r_f$$

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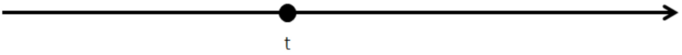
CAPM: $\hat{\beta}$ Sorted Portfolios



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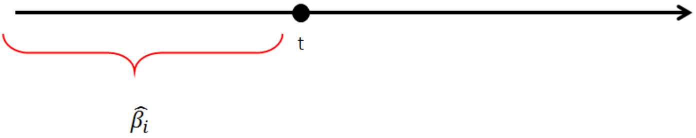

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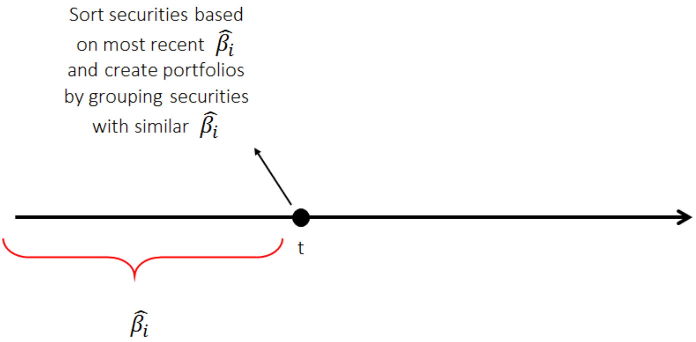

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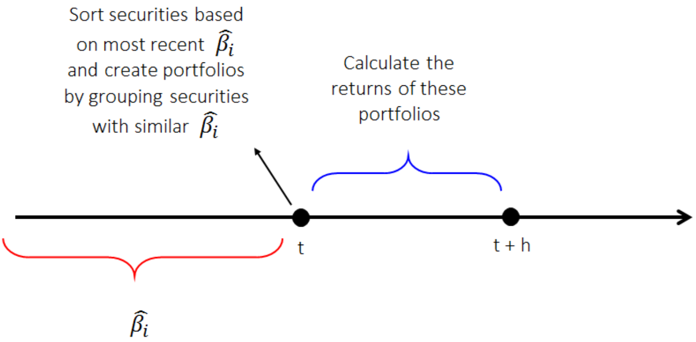

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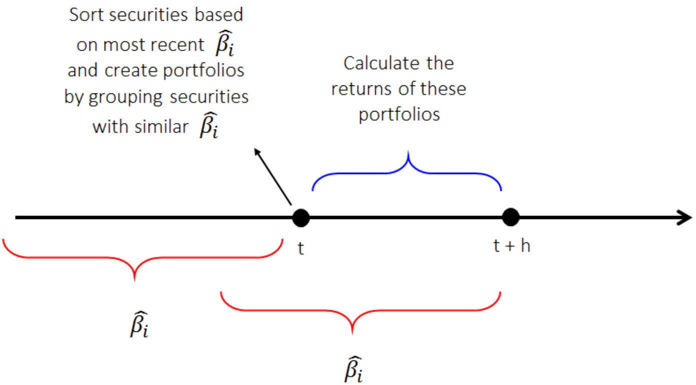

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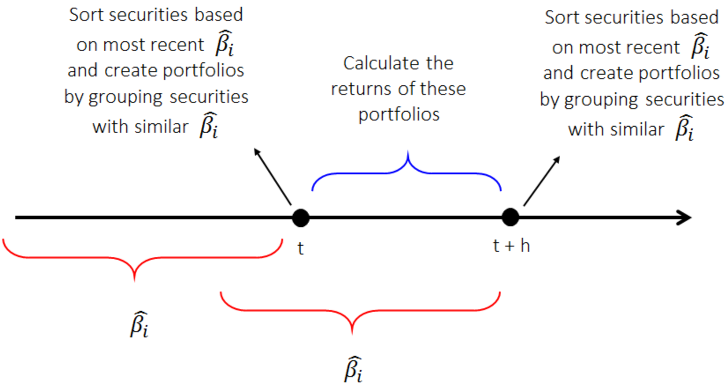

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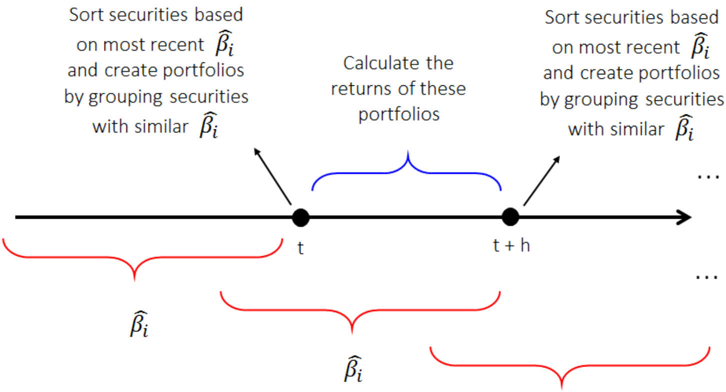
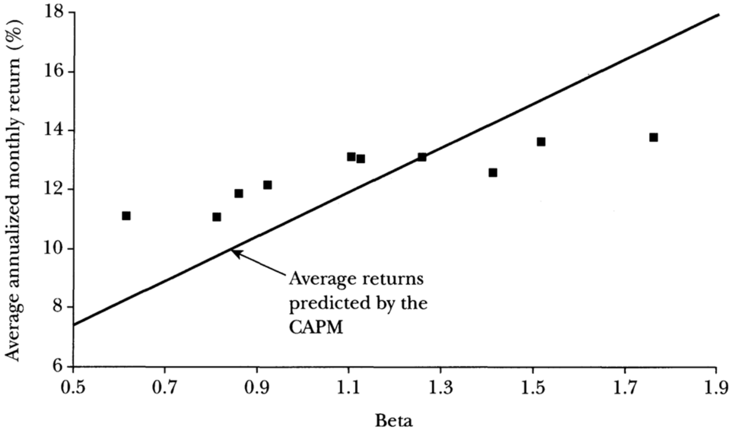


CAPM: SML from 1928 to 2003 (US Equity)



Source: Fama and French (2004) - *The Capital Asset Pricing Model: Theory and Evidence*

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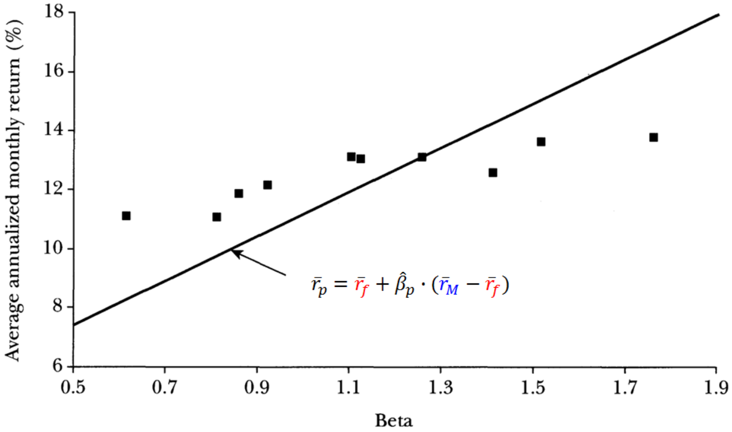

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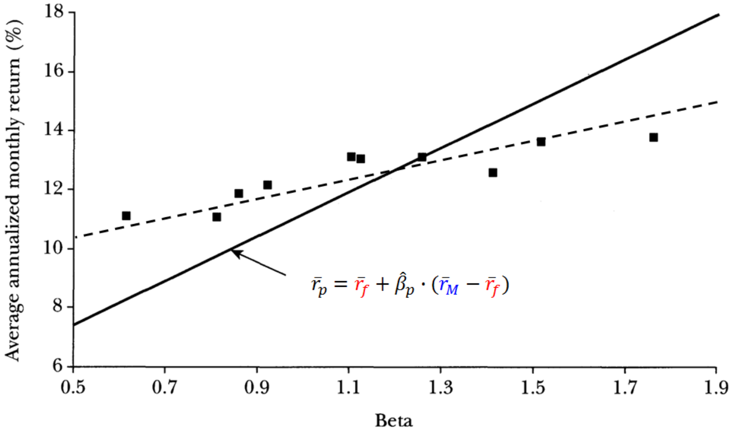

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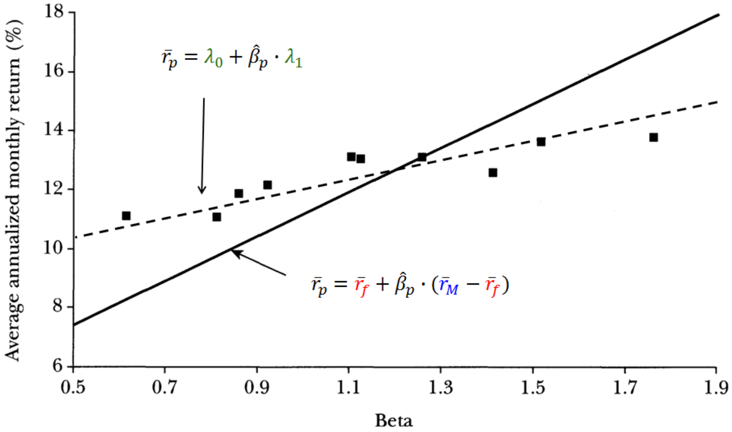

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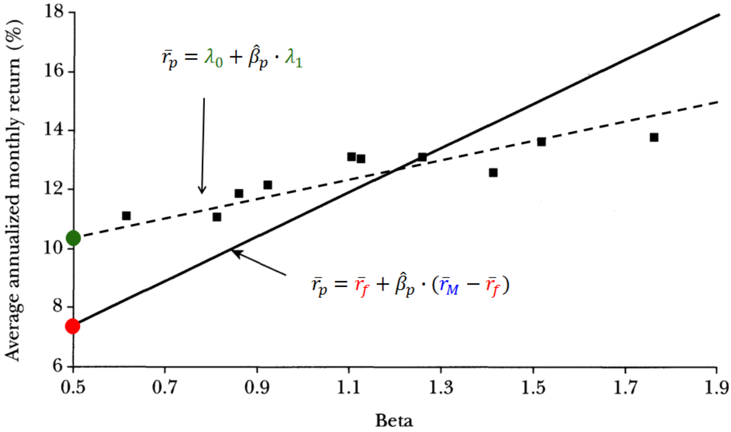

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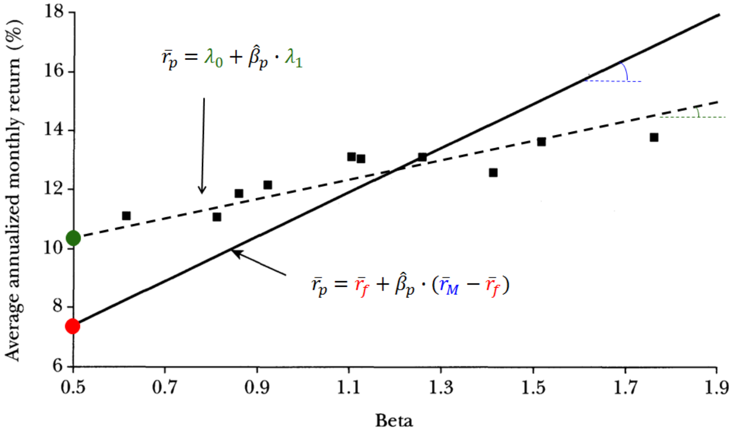

Source: Fama and French (2004) - *The Capital Asset Pricing Model: Theory and Evidence*

CAPM: α Prediction

- From CAPM, we have:

$$\mathbb{E}[r_i] - r_f = \beta_i \cdot (\mathbb{E}[r_M] - r_f)$$

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$$\Downarrow$$

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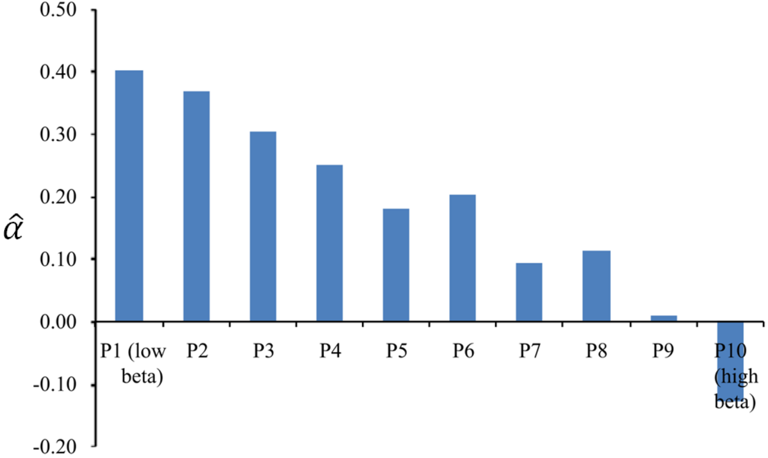
$$\Downarrow$$

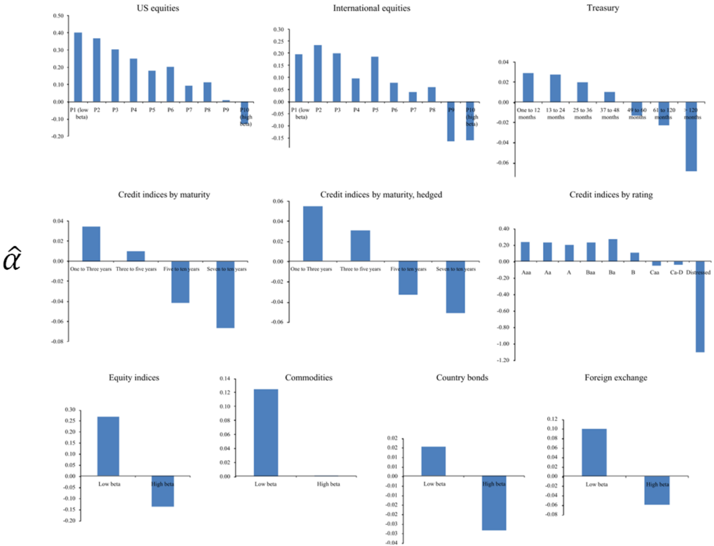
$$\alpha_i = 0$$

CAPM: $\hat{\alpha}$ from 1926 to 2012 (US Equity)



Source: Franzzini and Petersen (2014) - *Betting Against Beta*

CAPM: $\hat{\alpha}$ Across Asset Classes



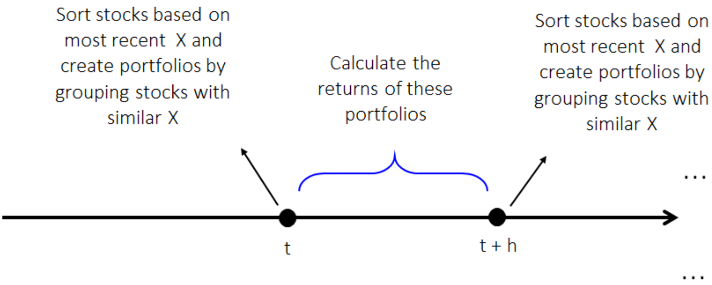
Source: Franzzini and Petersen (2014) - *Betting Against Beta*

CAPM: "β is the only Risk Measure" Prediction

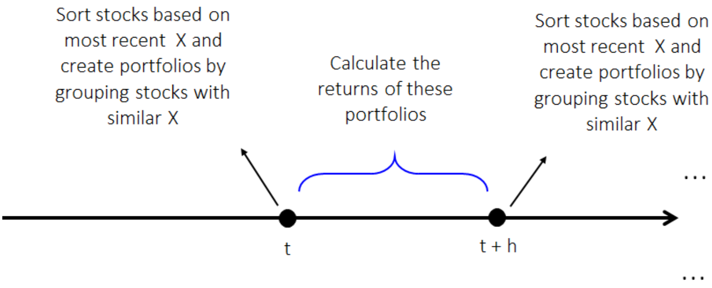
- From CAPM, we have:

$$\mathbb{E}[r_i] = r_f + \beta_i \cdot (\mathbb{E}[r_M] - r_f)$$

CAPM: Characteristic Sorted Portfolios



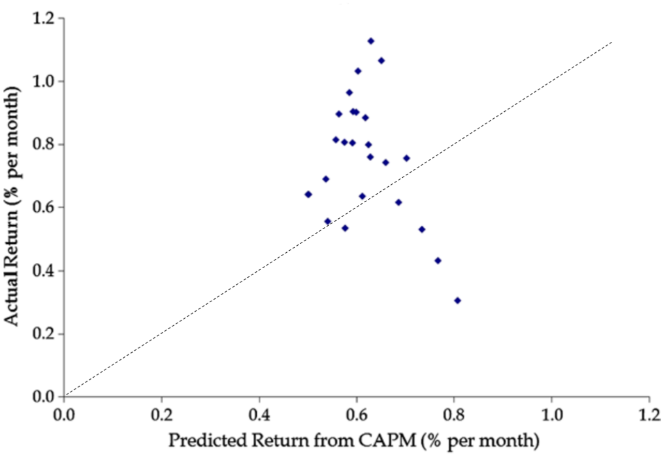
CAPM: Characteristic Sorted Portfolios



X = Size of Firm

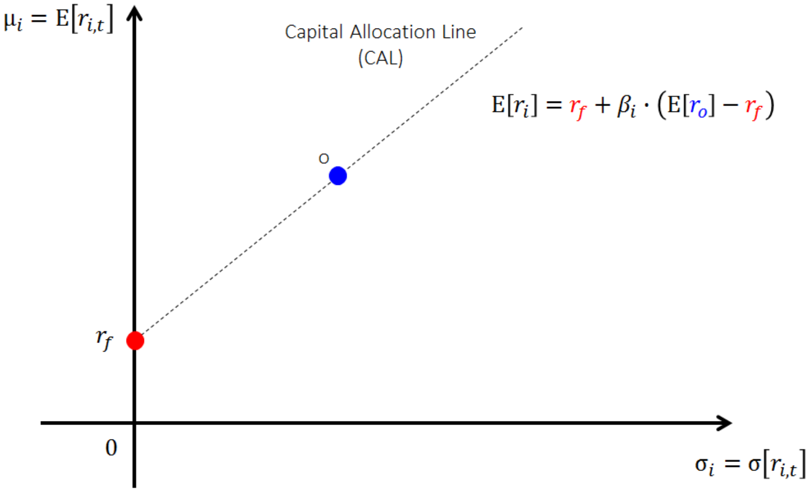
X = Book Equity / Market

CAPM: Size x Book-to-Market Portfolios (1946 to 2010)



Source: Goyal (2012) - *Empirical Cross-Sectional Asset Pricing: a Survey*

3 Factor Model: Equilibrium Justification



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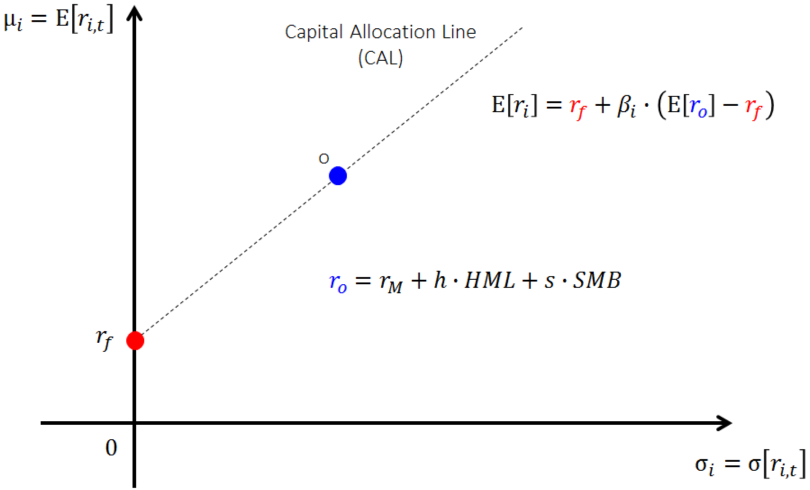

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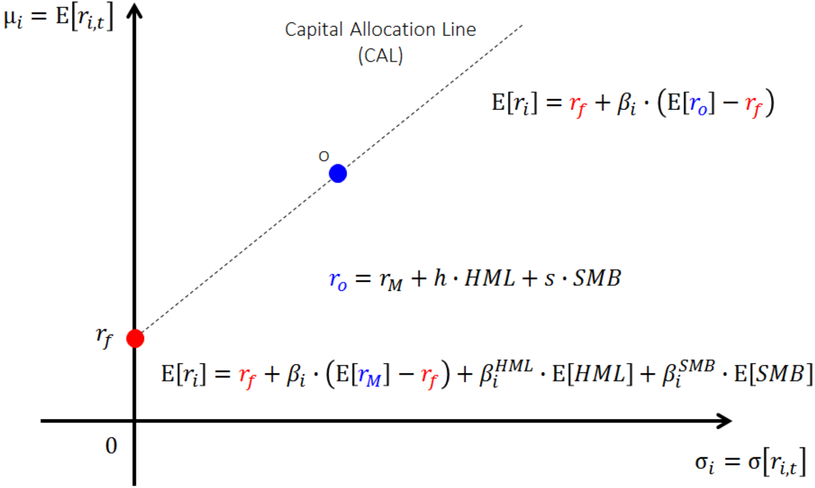

3 Factor Model: APT Justification

- Value stocks comove with other value stocks and growth stocks with other growth stocks. The same is true for small vs large companies. Therefore:

$$r_{i,t} - r_f = \alpha_i + \beta_i \cdot (r_{M,t} - r_f) + \beta_i^{HML} \cdot HML_t + \beta_i^{SMB} \cdot SMB_t + e_{i,t}$$

$$+$$

No free lunch in Wall street

$$\Downarrow$$

$$\mathbb{E}[r_p] = r_f + \beta_p \cdot (\mathbb{E}[r_M] - r_f) + \beta_p^{HML} \cdot \mathbb{E}[HML] + \beta_p^{SMB} \cdot \mathbb{E}[SMB]$$

3 Factor Model: APT Justification

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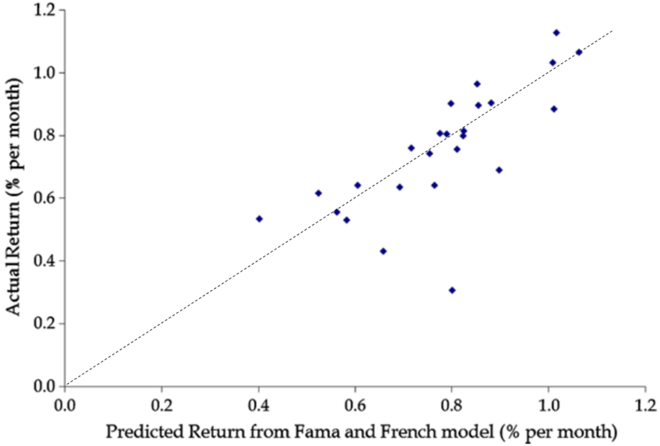
3 Factor Model: Size x Book-to-Market Porfolios (1946 to 2010)



Source: Goyal (2012) - *Empirical Cross-Sectional Asset Pricing: a Survey*

From an empirical perspective, there are three key failures of the CAPM. Can you explain what are the failures and what is the evidence behind each one of them?